CALCULUS I EXAM I SOLUTIONS

(I)

①. I. Let $\epsilon>0$ $\boxed{II}$ Search for δ III. Verify $\delta \Rightarrow \boxed{C}$

②. $\lim\limits_{x\to\pi} \tan(x) = \tan(\pi) = 0$ so NOT B; $\lim\limits_{x\to 3} \frac{x^2-9}{x+3} = \frac{(3)^2-9}{(3)+3} = \frac{0}{6}$ so NOT C;

$\lim\limits_{x\to 2} \frac{(x-2)}{(x^2-4)}$ Direct substitution gives $\frac{2-2}{(2)^2-4} \Rightarrow \frac{0}{0} \Rightarrow \boxed{D}$

③. $\lim\limits_{x\to 4^+} f(x) \overset{\text{BOTTOM}}{=} \lim\limits_{x\to 4^+} (2x-5) = 2(4)-5 = 3 \Rightarrow \boxed{C}$

④ (I) Let $\epsilon>0$ (II) Search for $\delta \Rightarrow |f(x)-L|<\epsilon \Rightarrow |(-6x-1)-(-13)|<\epsilon \Rightarrow$

$\Rightarrow |-6x-1+13|<\epsilon \Rightarrow |-6x+12|<\epsilon \Rightarrow |-6(x-2)|<\epsilon \Rightarrow |-6||x-2|<\epsilon$

$\Rightarrow |x-2| < \frac{\epsilon}{|-6|} \Rightarrow$ Choose $\delta = \frac{\epsilon}{|-6|} \Rightarrow \boxed{C}$

⑤. There is a discontinuity at $x=7$, Must check at $x=5$

(I) $f(5) \overset{\text{TOP}}{=} 2(5)-7 = 3$ ✓ II $\lim\limits_{x\to 5} f(x)$ ⓐ $\lim\limits_{x\to 5^-} f(x) \overset{\text{TOP}}{=} \lim\limits_{x\to 5^-} (2x-7) =$

$= 2(5)-7 = 3$ ✓ ⓑ $\lim\limits_{x\to 5^+} f(x) \overset{\text{BOT}}{=} \lim\limits_{x\to 5^+} \left(\frac{8}{x-7}\right) = \frac{8}{5-7} = -4$

Thus $\lim\limits_{x\to 5^-} f(x) \neq \lim\limits_{x\to 5^+} f(x) \Rightarrow \lim\limits_{x\to 5} f(x)$ D.N.E.

Thus there is a discontinuity at $x=5$. Thus there are <u>two</u> discontinuities $\Rightarrow \boxed{E}$

⑥. $\lim\limits_{x\to 5} f(x) = 8$ does NOT guarantee $f(5)$ exists thus

does not guarantee $f(x)$ is continuous at $x=5$ so, NOT B

$\lim\limits_{x\to 5} f(x) = 8$ does NOT guarantee $f(5)$ fails to exist, or

that $\lim\limits_{x\to 5} f(x) = f(5)$ so NOT C

→ continued

(#6) continued

$\lim_{x \to 5} f(x) = 8$ if and only if the left- and right-hand limits also equal 8 $\Rightarrow \boxed{D}$

(7) Same logic as #6 $\Rightarrow \boxed{B}$

(8) $0 < |x-a| < \delta$ assures $0 < |x-a| \Rightarrow x \neq a$ so $\boxed{B}$

(9) $\lim_{x \to \pi/2} f(x)$ Must check: $\lim_{x \to \pi/2^-} f(x) \overset{\text{TOP}}{=} \lim_{x \to \pi/2} (1+2x-\tan(2x)) =$

$= 1+2(\pi/2) - \tan(2(\pi/2)) = 1+\pi - \tan(\pi) = 1+\pi$

also $\lim_{x \to \pi/2^+} f(x) \overset{\text{BOT}}{=} \lim_{x \to \pi/2} (2x - \cos(2x)) = 2(\pi/2) - \cos(2(\pi/2)) =$

$= \pi - \cos(\pi) = \pi - (-1) = \pi + 1 = 1+\pi$

Thus $\lim_{x \to \pi/2} f(x) = 1+\pi \Rightarrow \boxed{E}$

(10). The limit's existence does not influence the function value, so NOT B. The right-hand limit does not influence the left-hand limit, so NOT C

If the one-sided limit D.N.E., then the unrestricted limit D.N.E. Thus, $\boxed{D}$

(11). $\lim_{x \to 9} \frac{\sqrt{x+7}-4}{x-9} = \lim_{x \to 9} \frac{(\sqrt{x+7}-4)(\sqrt{x+7}+4)}{(x-9)(\sqrt{x+7}+4)} = \lim_{x \to 9} \frac{x+7-16}{(x-9)(\sqrt{x+7}+4)} =$

CONTINUED

(11). continued

$$= \lim_{x\to 9} \frac{\cancel{(x-9)}}{\cancel{(x-9)}(\sqrt{x+7}+4)} = \lim_{x\to 9} \frac{1}{(\sqrt{x+7}+4)} = \frac{1}{\sqrt{9+7}+4} = \frac{1}{8} \to \boxed{B}$$

(12). There would be a discontinuity at $x=6$ but $x=6$ is not in the region of definition for the top rule. Hence no discontinuity at $x=6$. $\Rightarrow$ NOT A, NOT C, NOT D

There is a discontinuity at $x=5$

Must check $x=4$:

(I) $f(4) \overset{TOP}{=} 2-\frac{16}{(4-6)^2} = 2-\frac{16}{4} = -2$

(II) $\lim_{x\to 4} f(x) \Rightarrow$ (a) $\lim_{x\to 4^-} f(x) \overset{TOP}{=} 2-\frac{16}{(4-6)^2} = -2$

(b) $\lim_{x\to 4^+} f(x) \overset{BOT}{=} 2(4)+\frac{10}{4-5} = 8+(-10) = -2$

so $f(x)$ is continuous at $x=4$. $\Rightarrow$ NOT B $\Rightarrow \boxed{E}$

(13). The denominator of both $\sec(x)$ and $\tan(x)$ is $\cos(x)$ so there are discontinuities at the zeros of $\cos(x)$ that is, ODD MULTIPLES OF $\pi/2 \Rightarrow (2k+1)\pi/2 \Rightarrow \boxed{C}$

(14). All of B, C, and D are required for continuity $\Rightarrow \boxed{A}$

(15). Not A since in that case $\lim_{x\to 1^+} f(x) = 3$; Not B since in that case $\lim_{x\to 1^+} f(x) = 1$; In C $\lim_{x\to 1^+} f(x)$ D.N.E. $\Rightarrow \boxed{C}$.

II

16. For top rule $\Rightarrow x^2-4=0 \Rightarrow x=-2, x=2$, however omit $x=2$ since it does not satisfy $x \le 1 \Rightarrow \underline{x=-2}$ ✓

Bottom rule is a polynomial so no discontinuities.

Must check at $x=1$:

I) $f(1) \stackrel{\text{TOP}}{=} \frac{-3}{(1)^2-4} = \frac{-3}{-3} = 1$ ✓

II) $\lim_{x\to 1} f(x)$ @ $\lim_{x\to 1^-} f(x) \stackrel{\text{TOP}}{=} \lim_{x\to 1^-} \frac{-3}{x^2-4} = \frac{-3}{(1)^2-4} = \frac{-3}{-3} = 1$ ✓

(b) $\lim_{x\to 1^+} f(x) \stackrel{\text{BOT}}{=} \lim_{x\to 1^+} (-x^2+2) = -1+2 = 1$ ✓

so $\lim_{x\to 1} f(x) = 1$ ✓

III) so $\lim_{x\to 1} f(x) = f(1) \Rightarrow$ continuous at $x=1$

Thus $\underline{\text{ONLY}}$ Discontinuity is at $\boxed{x=-2}$

17. We must have $\lim_{x\to(-2)^-} f(x) = \lim_{x\to(-2)^+} f(x)$

Thus $\lim_{x\to(-2)^-} (x^2-kx) = \lim_{x\to(-2)^+} (4x+2) \Rightarrow (-2)^2-k(-2) = 4(-2)+2 \Rightarrow$

$\Rightarrow 4+2k=-6 \Rightarrow \boxed{k=-5}$

18. $$\lim_{\Delta x\to 0} \frac{(x+\Delta x)^2-3(x+\Delta x)+2-(x^2-3x+2)}{\Delta x} =$$

$$= \lim_{\Delta x\to 0} \frac{\cancel{x^2}+2x\Delta x+\Delta x^2-\cancel{3x}-3\Delta x+\cancel{2}-\cancel{x^2}+\cancel{3x}-\cancel{2}}{\Delta x} = \lim_{\Delta x\to 0} \frac{2x\Delta x+\Delta x^2-3\Delta x}{\Delta x} =$$

$$= \lim_{\Delta x\to 0} \frac{\cancel{\Delta x}(2x+\Delta x-3)}{\cancel{\Delta x}} = \lim_{\Delta x\to 0} (2x+\Delta x-3) = 2x+0-3 = \boxed{2x-3}$$

19. $\lim\limits_{x\to 3} \frac{(\sqrt{x+1}-2)}{(x-3)} = \lim\limits_{x\to 3} \frac{(\sqrt{x+1}-2)(\sqrt{x+1}+2)}{(x-3)(\sqrt{x+1}+2)} =$ pg 5 five

$= \lim\limits_{x\to 3} \frac{x+1-4}{(x-3)(\sqrt{x+1}+2)} = \lim\limits_{x\to 3} \frac{\cancel{(x-3)}}{\cancel{(x-3)}(\sqrt{x+1}+2)} =$

$= \lim\limits_{x\to 3} \frac{1}{(\sqrt{x+1}+2)} = \frac{1}{\sqrt{3+1}+2} = \boxed{\frac{1}{4}}$

20. Top rule has no discontinuities. the bottom rule IS ONLY IN EFFECT FOR $x>0$ so there ARE discontinuities at $x=\pi/2$ and $x=3\pi/2$ BUT NOT AT $x=-\pi/2$ and $x=\frac{-3\pi}{2}$.

Must check at $x=0$

II $\lim\limits_{x\to 0^-} f(x) \overset{TOP}{=} \lim\limits_{x\to 0^-} (-\cos(x)) = -\cos(0) = -1$

$\lim\limits_{x\to 0^+} f(x) \overset{BUT}{=} \lim\limits_{x\to 0^+} \tan(x) = \tan(0) = 0$

$\Rightarrow$ Thus $\lim\limits_{x\to 0} f(x)$ D.N.E

Thus Discontinuity at $x=0$ $\Rightarrow$ $\boxed{x=0,\ x=\pi/2,\ x=3\pi/2}$

21. $$\lim_{x\to 4} \frac{x^2-6x+8}{x^2-x-12} = \lim_{x\to 4} \frac{\cancel{(x-4)}(x-2)}{(x+3)\cancel{(x-4)}}$$

$$= \lim_{x\to 4} \frac{(x-2)}{(x+3)}$$

$$= \frac{4-2}{4+3} = \boxed{\frac{2}{7}}$$

22. $$\lim_{x\to 0} \frac{2\tan(3x)}{x} = \lim_{x\to 0} \frac{2\sin(3x)}{x\cdot\cos(3x)}$$

$$= \lim_{x\to 0} \frac{3\cdot 2\sin(3x)}{3\cdot x\cos(3x)}$$

$$= \lim_{x\to 0} \frac{\sin(3x)}{3x}\cdot\frac{6}{\cos(3x)}$$

$$= 1\cdot\frac{6}{\cos(3(0))} = 1\cdot\frac{6}{1} = \boxed{6}$$

23. $$\lim_{x\to 9} \frac{(\sqrt{18-x}-\sqrt{x})}{(9-x)} = \lim_{x\to 9} \frac{(\sqrt{18-x}-\sqrt{x})(\sqrt{18-x}+\sqrt{x})}{(9-x)(\sqrt{18-x}+\sqrt{x})}$$

$$= \lim_{x\to 9} \frac{(18-x-x)}{(9-x)(\sqrt{18-x}+\sqrt{x})}$$

$$= \lim_{x\to 9} \frac{(18-2x)}{(9-x)(\sqrt{18-x}+\sqrt{x})}$$

$$= \lim_{x\to 9} \frac{2\cancel{(9-x)}}{\cancel{(9-x)}(\sqrt{18-x}+\sqrt{x})}$$

$$= \frac{2}{\sqrt{18-9}+\sqrt{9}}$$

$$= \frac{2}{\sqrt{9}+\sqrt{9}}$$

$$= \frac{2}{6} = \boxed{\frac{1}{3}}$$

(24). $\lim\limits_{x \to (-1)} f(x)$ where $f(x) = \begin{cases} 4x - x^2 & x < -1 \\ \frac{15}{x+4} & x > -1 \end{cases}$

(a) $\lim\limits_{x \to (-1)^-} f(x) \overset{\text{TOP}}{=} \lim\limits_{x \to (-1)^-} (4x - x^2) = 4(-1) - (-1)^2 = -4 - 1 = -5$

(b) $\lim\limits_{x \to (-1)^+} f(x) \overset{\text{BOT}}{=} \lim\limits_{x \to (-1)^+} \left(\frac{15}{x+4}\right) = \frac{15}{(-1)+4} = \frac{15}{3} = 5$

Thus $\boxed{\lim\limits_{x \to (-1)} f(x) \text{ D.N.E.}}$

(25) $\lim\limits_{x \to 5} \frac{(3-x)}{(x-5)}$ $\quad \lim\limits_{x \to 5^-} \frac{(3-x)}{(x-5)}$ if $x = 4.9 \to \frac{3-4.9}{4.9-5} = \frac{NEG}{NEG} \to pos \to +\infty$

$\lim\limits_{x \to 5^+} \frac{(3-x)}{(x-5)}$ if $x = 5.1$ $\frac{3-5.1}{5.1-5} = \frac{NEG}{POS} \to NEG \to -\infty$

$\boxed{\therefore \lim\limits_{x \to 5} \frac{3-x}{x-5} \text{ D.N.E.}}$

(26). Prove $\lim\limits_{x \to (-3)} (-5x - 6) = 9$

(I) Let $\epsilon > 0$

(II) Search for δ:

we want: $|f(x) - L| < \epsilon$

$\Rightarrow |(-5x-6) - 9| < \epsilon$

$\Rightarrow |-5x - 15| < \epsilon$

$\Rightarrow |-5(x+3)| < \epsilon$

$\Rightarrow |-5|\,|x - (-3)| < \epsilon$

$\Rightarrow |x - (-3)| < \frac{\epsilon}{|-5|}$

Choose $\delta = \frac{\epsilon}{|-5|}$

(III) Verify δ:

$0 < |x - a| < \delta \Rightarrow$

$\Rightarrow |x - (-3)| < \frac{\epsilon}{|-5|}$

$\Rightarrow |-5|\,|x+3| < \epsilon$

$\Rightarrow |-5x - 15| < \epsilon$

$\Rightarrow |-5x - 6 + 6 - 15| < \epsilon$

$\Rightarrow |(-5x-6) - 9| < \epsilon$

$\Rightarrow \boxed{|f(x) - L| < \epsilon}$