SOLUTIONS
PRACTICE EXAM I

I

(1) $\lim_{x \to 2\pi} \frac{5-\cos(x)}{\sin(x)+2} = \frac{5-\cos(2\pi)}{\sin(2\pi)+2}$ substitute and simplify

$= \frac{5-1}{0+2}$

$= \frac{4}{2} = \boxed{2}$

(2) $\lim_{x \to (-3)} \frac{x^2+8x+15}{x^2-x-12} = \lim_{x \to (-3)} \frac{\cancel{(x+3)}(x+5)}{(x-4)\cancel{(x+3)}}$

$= \lim_{x \to (-3)} \frac{(x+5)}{(x-4)}$

$= \frac{-3+5}{-3-4}$

$= \boxed{-\frac{2}{7}}$

(3) $\lim_{h \to 0} \frac{2(a+h)^2+4(a+h)-5-(2a^2+4a-5)}{h} =$

$= \lim_{h \to 0} \frac{2(a^2+2ah+h^2)+4a+4h-5-2a^2-4a+5}{h}$

$= \lim_{h \to 0} \frac{\cancel{2a^2}+4ah+2h^2+\cancel{4a}+4h\cancel{-5}\cancel{-2a^2}\cancel{-4a}\cancel{+5}}{h}$

$= \lim_{h \to 0} \frac{4ah+2h^2+4h}{h}$

$= \lim_{h \to 0} \frac{\cancel{h}(4a+2h+4)}{\cancel{h}} = \lim_{h \to 0} (4a+2h+4) = 4a+2(0)+4$

$= \boxed{4a+4}$

④ $\lim_{x\to 2}\left(3x^3-5x^2-6|x-1|\right) = 3(2)^3-5(2)^2-6|2-1|$

$= 24-20-6$

$= \boxed{-2}$

⑤ $\lim_{x\to 0}\dfrac{\sin(8x)\tan(x)}{2x^2} = \lim_{x\to 0}\dfrac{\sin(8x)\cdot\tan(x)}{2x\cdot x}$

$= \lim_{x\to 0}\dfrac{\sin(8x)}{2x}\cdot\lim_{x\to 0}\dfrac{\tan(x)}{x}$

$= \lim_{x\to 0}\dfrac{4\cdot\sin(8x)}{4\cdot 2x}\cdot\lim_{x\to 0}\dfrac{\sin(x)}{x\cdot\cos(x)}$

$= 4\lim_{x\to 0}\dfrac{\sin(8x)}{8x}\cdot\lim_{x\to 0}\dfrac{\sin(x)}{x}\cdot\lim_{x\to 0}\dfrac{1}{\cos(x)}$

$= 4(1)\cdot(1)\cdot\dfrac{1}{\cos(0)}$

$= 4\cdot 1\cdot 1$

$= \boxed{4}$

⑥ $\lim_{x\to 0}\dfrac{\tan(x)}{x\cdot\sec(x)} = \lim_{x\to 0}\dfrac{\sin(x)\cdot\cos(x)}{\cos(x)\cdot x}$

$= \lim_{x\to 0}\dfrac{\sin(x)}{x}$

$= \boxed{1}$

7. $$\lim_{x\to(-5)} \frac{(\sqrt{x+14}-3)}{(x+5)} = \lim_{x\to(-5)} \frac{(\sqrt{x+14}-3)}{(x+5)} \cdot \frac{(\sqrt{x+14}+3)}{(\sqrt{x+14}+3)}$$

$$= \lim_{x\to(-5)} \frac{(\sqrt{x+14})^2 - 9}{(x+5)(\sqrt{x+14}+3)}$$

$$= \lim_{x\to(-5)} \frac{x+14-9}{(x+5)(\sqrt{x+14}+3)}$$

$$= \lim_{x\to(-5)} \frac{\cancel{(x+5)}}{\cancel{(x+5)}(\sqrt{x+14}+3)}$$

$$= \lim_{x\to(-5)} \frac{1}{\sqrt{x+14}+3}$$

$$= \frac{1}{\sqrt{(-5)+14}+3}$$

$$= \frac{1}{3+3} = \boxed{\frac{1}{6}}$$

8. $$\lim_{x\to 4^+} \frac{x+8}{x-4}$$ there is an asymptote at $x=4$

so $x \to 4^+ \Rightarrow x+8 \to$ "pos"

$x-4 \to$ "pos"

so $$\lim_{x\to 4^+} \frac{x+8}{x-4} \to \frac{\text{"pos"}}{\text{"pos"}} \Rightarrow \boxed{+\infty}$$

9. $\lim_{x\to 2} f(x)$ where $f(x) = \begin{cases} 4x - x^2, & x < 2 \\ \dfrac{18}{x+4}, & x \geq 2 \end{cases}$

Note that $x=2$ is where the function changes behavior:

so

$$\lim_{x\to 2^-} f(x) \overset{\text{top}}{=\!=} \lim_{x\to 2^-} (4x - x^2)$$

$$= 4(2) - (2)^2$$

$$= 8 - 4$$

$$= 4$$

and

$$\lim_{x\to 2^+} f(x) \overset{\text{bottom}}{=\!=} \lim_{x\to 2^-} \left(\frac{18}{x+4}\right)$$

$$= \frac{18}{2+4}$$

$$= \frac{18}{6}$$

$$= 3$$

since the left- and right-hand limits do not agree, the unrestricted limits

$\lim_{x\to 2} f(x)$ $\boxed{\text{DOES NOT EXIST}}$

(10). $\lim_{x \to 5} f(x)$ where $f(x) = \begin{cases} x^2 - 1 & \text{if } x \le 2 \\ 3x-4 & \text{if } x > 2 \end{cases}$ $\underline{\text{5 is ve}}$

note that $x=5$ is in the interval $x>2$

so $\lim_{x\to 5} f(x) \overset{\text{bottom}}{=\!=} \lim_{x\to 5}(3x-4)$

$= 3(5)-4$

$= \boxed{11}$

(11). $\lim_{x\to\pi} f(x)$ where $f(x) = \begin{cases} \tan(x) - 3\cos(x) & \text{if } x < \pi \\ \dfrac{6}{\sin(x)+2} & \text{if } x \ge \pi \end{cases}$

must check

$\lim_{x\to\pi^-} f(x) \overset{\text{top}}{=} \lim_{x\to\pi^-}(\tan(x) - 3\cos(x))$

$= \tan(\pi) - 3\cos(\pi)$

$= 0 - 3(1)$

$= -3$ ✓

$\lim_{x\to\pi^+} f(x) \overset{\text{bottom}}{=\!=} \lim_{x\to\pi^+}\left(\frac{6}{\sin(x)+2}\right)$

$= \dfrac{6}{\sin(\pi)+2}$

$= \dfrac{6}{2}$

$= 3$ ✓

However, since these do not agree, the unrestricted limit $\lim_{x\to\pi} f(x)$ $\boxed{\text{D.N.E.}}$

(12). $\lim_{x\to 3^-}\left(\frac{-5x}{3-x}\right)$ there is an asymptote at $x=3$

so

$x \to 3^- \Rightarrow$ $-5x \to$ "NEG"

$3-x \to$ "pos"

so $\lim_{x\to 3^-}\left(\frac{-5x}{3-x}\right) \to \frac{\text{"NEG"}}{\text{"pos"}} \to$ NEG

$\Rightarrow \boxed{-\infty}$

II (13) $f(x) = \frac{x^3}{x^3-9x}$ Not continuous if $x^3-9x=0$

$\Rightarrow x(x^2-9)=0$

$\Rightarrow x=0$ or $x=\pm 3$

NOT CONTINUOUS
AT $x=0$, $x=-3$, $x=3$

(14) $f(x)=\sec(x)$ NOT CONTINUOUS IF $\cos(x)=0$

$= \frac{1}{\cos(x)}$ $\Rightarrow$ NOT CONTINUOUS at

ODD MULTIPLES OF $\frac{\pi}{2}$

OR

$\boxed{(2k+1)\left(\frac{\pi}{2}\right)}$

(15) $f(x) = \begin{cases} \frac{8x}{x^2+1} & x \le -1 \\ \frac{4}{x} & x > 1 \end{cases}$

NOT CONTINUOUS (a) IF $x^2+1=0$

$\Rightarrow x^2 = -1$

$\Rightarrow x = \pm\sqrt{-1}$ NO SOLUTION, SO NO DISCONTINUITIES HERE

(b) IF $x = 0$

MUST CHECK AT $x=-1$

(i) $f(-1) = \frac{8(-1)}{(-1)^2+1} = \frac{-8}{2} = -4$

(ii) $\lim_{x\to(-1)^-} f(x) \underset{=}{\text{top}} \lim_{x\to(-1)^-} \left(\frac{8x}{x^2+1}\right) = \frac{8(-1)}{(-1)^2+1} = \frac{-8}{2} = -4$

$\lim_{x\to(-1)^+} f(x) \underset{=}{\text{bottom}} \lim_{x\to(-1)^+} \left(\frac{4}{x}\right) = \frac{4}{-1} = -4$

since these equal, $\lim_{x\to 2} f(x) = -4$

(iii) since the answers for (i) and (ii) are equal, we have $f(x)$ is continuous at $x=1$

$\therefore$ $\boxed{\text{The only DISCONTINUITIE IS AT } x=0}$

(16). $f(x) = \begin{cases} -\cos(x) & \text{if } x \leq \pi \\ x & \text{if } x > \pi \end{cases}$

Neither $-\cos(x)$ nor x have discontinuities so the only possibility is at $x = \pi$, which we must check:

(i) $f(\pi) \overset{\text{top}}{=} -\cos(\pi) = -1$

(ii) $\lim_{x \to \pi^-} f(x) \overset{\text{TOP}}{=} \lim_{x \to \pi^-} (-\cos(x))$

$= -\cos(\pi)$

$= -(-1)$

$= 1$

$\lim_{x \to \pi^+} f(x) \overset{\text{bottom}}{=} \lim_{x \to \pi^+} (x)$

$= \pi$

Since the left- and right-hand limits do not equal, $\lim_{x \to \pi} f(x)$ D.N.E.

$\therefore$ $\boxed{\text{NOT CONTINUOUS AT } x = \pi}$

III (17). $f(x) = \begin{cases} x^2 + 4x + 2 & \text{if } x < 1 \\ kx + 3 & \text{if } x \geq 1 \end{cases}$

we require $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x)$

$\Rightarrow \overset{\text{TOP}}{\lim_{x \to 1^-}} (x^2 + 4x + 2) = \overset{\text{Bottom}}{\lim_{x \to 1^+}} (kx + 3)$

$\Rightarrow (1)^2 + 4(1) + 2 = k(1) + 3$

$\Rightarrow 7 = k + 3 \Longrightarrow \boxed{k = 4}$

(18) $f(x) = \begin{cases} x^2 - \sin(x) + k_1 & \text{if } x < \pi \\ x^2 - \cos(x) - 2 & \text{if } \pi \le x < 2\pi \\ x^2 - \tan(x) + k_2 & \text{if } x \ge 2\pi \end{cases}$

We require

$$\lim_{x \to \pi^-} f(x) = \lim_{x \to \pi^+} f(x) \quad \underline{\text{AND}} \quad \lim_{x \to 2\pi^-} f(x) = \lim_{x \to 2\pi^+} f(x)$$

$$\Rightarrow \lim_{x \to \pi^-} \overbrace{\left(x^2 - \sin(x) + k_1\right)}^{\text{TOP}} = \lim_{x \to \pi^+} \overbrace{\left(x^2 - \cos(x) - 2\right)}^{\text{MIDDLE}}$$

$$\Rightarrow \pi^2 - \sin(\pi) + k_1 = \pi^2 - \cos(\pi) - 2)$$

$$\Rightarrow \cancel{\pi^2} + k_1 = \cancel{\pi^2} - (-1) - 2$$

$$\Rightarrow k_1 = -1$$

$$\underline{\text{AND}} \lim_{x \to 2\pi^-} \overbrace{\left(x^2 - \cos(x) - 2\right)}^{\text{MIDDLE}} = \lim_{x \to 2\pi^+} \overbrace{\left(x^2 - \tan(x) + k_2\right)}^{\text{bottom}}$$

$$\Rightarrow (2\pi)^2 - \cos(2\pi) - 2 = (2\pi)^2 - \tan(2\pi) + k_2$$

$$\Rightarrow \cancel{4\pi^2} - 1 - 2 = \cancel{4\pi^2} - 0 + k_2$$

$$\Rightarrow \boxed{-3 = k_2}$$

III

(19) Prove $\lim_{x \to 2} (4x-1) = 7$

(i) Let $\epsilon > 0$

(ii) Search for δ:

We want $|f(x) - L| < \epsilon$

$\Rightarrow |(4x-1) - 7| < \epsilon$

$\Rightarrow |4x - 8| < \epsilon$

$\Rightarrow |4(x-2)| < \epsilon$

$= |4||x-2| < \epsilon$

$\Rightarrow |x-2| < \frac{\epsilon}{4}$

Choose $\delta = \epsilon/4$

(iii) Verify this δ:

$0 < |x-a| < \delta \Rightarrow |x-2| < \frac{\epsilon}{4}$

$\Rightarrow 4|x-2| < \epsilon$

$\Rightarrow |4x-8| < \epsilon$

$\Rightarrow |4x-1+1-8| < \epsilon$

$\Rightarrow |(4x-1) - 7| < \epsilon$

$\Rightarrow |f(x) - L| < \epsilon$ ✓

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(20) Prove $\lim_{x \to (-2)} (-3x+4) = 10$

(i) Let $\epsilon > 0$

(ii) Search for δ

We want $|f(x) - L| < \epsilon$

$\Rightarrow |(-3x+4) - 10| < \epsilon$

$\Rightarrow |-3x-6| < \epsilon$

$\Rightarrow |-3(x+2)| < \epsilon$

$\Rightarrow |-3||x-(-2)| < \epsilon$

$\Rightarrow |x-(-2)| < \frac{\epsilon}{|-3|} = \frac{\epsilon}{3}$

Choose $\delta = \frac{\epsilon}{3}$

(iii) Verify this δ

- $0 < |x-a| < \delta \Rightarrow |x-(-2)| < \frac{\epsilon}{3}$

$\Rightarrow |x+2| < \frac{\epsilon}{3}$

$\Rightarrow 3|x+2| < \epsilon$

$\Rightarrow |-3||x+2| < \epsilon$

$\Rightarrow |-3x-6| < \epsilon$

$\Rightarrow |-3x+4-4-6| < \epsilon$

$\Rightarrow |(-3x+4) - 10| < \epsilon$

$\Rightarrow |f(x) - L| < \epsilon$ ✓