SOLUTIONS CALC I

PRACTICE EXAM I

I.

(1) $\lim_{x\to 2\pi} \frac{5-\cos(x)}{\sin(x)+2} = \frac{5-\cos(2\pi)}{\sin(2\pi)+2} = \frac{5-1}{0+2} = \boxed{2}$

(2) $\lim_{x\to(-3)} \frac{x^2+8x+15}{x^2-x-12} = \lim_{x\to(-3)} \frac{(x+5)(x+3)}{(x-4)(x+3)}$

$= \lim_{x\to(-3)} \frac{(x+5)}{(x-4)}$

$= \frac{-3+5}{-3-4}$

$= \boxed{\frac{2}{-7}}$

(3) $\lim_{h\to 0} \frac{2(a+h)^2+4(a+h)-5-(2a^2+4a-5)}{h} =$

$= \lim_{h\to 0} \frac{2(a^2+2ah+h^2)+4(a+h)-5-(2a^2+4a-5)}{h}$

$= \lim_{h\to 0} \frac{\cancel{2a^2}+4ah+2h^2+\cancel{4a}+4h\cancel{-5}\cancel{-2a^2}\cancel{-4a}\cancel{+5}}{h}$

$= \lim_{h\to 0} \frac{4ah+2h^2+4h}{h}$

$= \lim_{h\to 0} \frac{h(4a+2h+4)}{h}$

$= \lim_{h\to 0} (4a+2h+4)$

$= 4a+2(0)+4$

$= \boxed{4a+4}$

(4) $\lim_{x\to 2} (3x^3 - 5x^2 - 6|x-1|) =$

$= 3(2)^3 - 5(2)^2 - 6|2-1|$

$= 3(8) - 5(4) - 6(1)$

$= 24 - 20 - 6$

$= \boxed{-2}$

(5) $\lim_{x\to 0} \frac{\sin(8x)\tan(x)}{2x} = \lim_{x\to 0} \frac{4\sin(8x)\tan(x)}{4\cdot 2x}$

$= \lim_{x\to 0} \frac{\sin(8x)}{8x} \cdot 4\tan(x)$

$= \lim_{x\to 0} \frac{\sin(8x)}{8x} \cdot \lim_{x\to 0} 4\tan(x)$

$= 1 \cdot 4(0)$

$= \boxed{0}$

(oops: I copied the problem incorrectly. See my ammended solution at the top of pg 3)

(6) $\lim_{x\to 0} \frac{\tan(x)}{x\sec(x)} = \lim_{x\to 0} \frac{\sin(x)}{\cos(x)\cdot x\sec(x)}$

$= \lim_{x\to 0} \frac{\sin(x)}{x} \cdot \lim_{x\to 0} \frac{1}{\cos(x)\sec(x)}$

$= \lim_{x\to 0} \frac{\sin(x)}{x} \cdot \lim_{x\to 0} \frac{\cos(x)}{\cos(x)}$

$= (1)\cdot\left(\frac{1}{1}\right)$

$= \boxed{1}$

(5). NOTE: This is the solution for the CORRECTLY COPIED item (#5).

$$\lim_{x\to 0} \frac{\sin(8x)\tan(x)}{2x^2} = \lim_{x\to 0} \frac{4\sin(8x)\tan(x)}{8x\cdot x}$$

$$= \lim_{x\to 0} \frac{\sin(8x)}{8x} \cdot \lim_{x\to 0} \frac{4\tan(x)}{x}$$

$$= \lim_{x\to 0} \frac{\sin(8x)}{8x} \cdot \lim_{x\to 0} \frac{4\sin(x)}{x\cdot\cos(x)}$$

$$= \lim_{x\to 0} \frac{\sin(8x)}{8x} \cdot \lim_{x\to 0} \frac{\sin(x)}{x} \cdot \lim_{x\to 0} \frac{4}{\cos(x)}$$

$$= (1)\cdot(1)\cdot\left(\frac{4}{1}\right)$$

$$= \boxed{4}.$$

Now, continuing on---

(7) $$\lim_{x\to(-5)} \frac{(\sqrt{x+14}-3)}{(x+5)} = \lim_{x\to(-5)} \frac{(\sqrt{x+14}-3)}{(x+5)} \frac{(\sqrt{x+14}+3)}{(\sqrt{x+14}+3)}$$

$$= \lim_{x\to(-5)} \frac{(x+14-9)}{(x+5)(\sqrt{x+14}+3)}$$

$$= \lim_{x\to(-5)} \frac{\cancel{(x+5)}}{\cancel{(x+5)}(\sqrt{x+14}+3)}$$

$$= \lim_{x\to(-5)} \frac{1}{(\sqrt{x+14}+3)}$$

$$= \frac{1}{\sqrt{(-5)+14}+3} = \frac{1}{\sqrt{9}+3} = \boxed{\frac{1}{6}}$$

8. $\lim_{x \to 4^+} \frac{x+8}{x-4}$ note direct substitution.

goes to $\frac{4+8}{4-4} \Rightarrow \frac{12}{0} \to \frac{\text{NON-ZERO}}{\text{ZERO}}$

so we use the special scenario technique.

$\Rightarrow$ If $x = 4.1 \Rightarrow \frac{4.1+8}{4.1-4} = \frac{\text{pos}}{\text{pos}} \Rightarrow +\infty$

so $\lim_{x \to 4^+} \frac{x+8}{x-4} = \boxed{+\infty}$.

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9. $f(x) = \begin{cases} 4x - x^2 & , x < 2 \\ \frac{18}{x+4} & , x \geq 2 \end{cases}$

$\lim_{x \to 2} f(x)$: since $x = 2$ is the value where the function changes behavior we must check the left- and right-hand limits

ⓐ $\lim_{x \to 2^-} f(x) \overset{\text{TOP}}{=} \lim_{x \to 2^-} (4x - x^2) = 4(2) - (2)^2 = 4$ ←

ⓑ $\lim_{x \to 2^+} f(x) \overset{\text{BOT}}{=} \lim_{x \to 2^+} \frac{18}{x+4} = \frac{18}{(2)+4} = 3$ ←

Since these do NOT agree:

$\boxed{\lim_{x \to 2} f(x) \text{ D.N.E.}}$

(10). $f(x) = \begin{cases} x^2 - 1, & x \le 2 \\ 3x - 4, & x > 2 \end{cases}$

$\lim_{x \to 5} f(x)$ since the function does not change behavior at $x = 5$, we may simply substitute and simplify:

VIZ:

$$\lim_{x \to 5} f(x) \stackrel{BUT}{=} \lim_{x \to 5} (3x - 4) = 3(5) - 4 = \boxed{11}$$

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(11). $\lim_{x \to \pi} f(x)$ $\quad f(x) = \begin{cases} \tan(x) - 3\cos(x), & x < \pi \\ \dfrac{6}{\sin(x) + 2}, & x \ge \pi \end{cases}$

must check left- and right-hand limits:

$$@ \lim_{x \to \pi^-} f(x) \stackrel{TOP}{=} \lim_{x \to \pi^-} (\tan(x) - 3\cos(x)) = \tan(\pi) - 3\cos(\pi)$$

$$= 0 - 3(-1) = 3$$

$$⑥ \lim_{x \to \pi^+} f(x) \stackrel{BOT}{=} \lim_{x \to \pi^+} \frac{6}{\sin(x) + 2} = \frac{6}{\sin(\pi) + 2} = \frac{6}{2} = 3$$

These AGREE

Thus $\lim_{x \to \pi} f(x) = \boxed{3}$

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(12) $\lim_{x \to 3^-} \dfrac{-5x}{3 - x}$ Direct substitution gives $\dfrac{\text{NON-ZERO}}{\text{ZERO}}$

so we use the special process:

If $x = 2.9 \Rightarrow \dfrac{-5(2.5)}{3 - (2.9)} \Rightarrow \dfrac{NEG}{POS} \Rightarrow -\infty$

So $\lim_{x \to 3^-} \dfrac{-5x}{3 - x} = \boxed{-\infty}$.

II. (13) $f(x) = \dfrac{x^3}{x^3 - 9x}$

Not continuous if Denominator $= 0$

$\Rightarrow$ If $x^3 - 9x = 0 \Rightarrow x(x^2 - 9) = 0$

$\Rightarrow x(x+2)(x-2) = 0$

$\Rightarrow x = 0$; $x = -2$; $x = 2$

So $f(x)$ is NOT continuous at $\boxed{x = 0,\ x = -2,\ x = 2}$

NOTE: If you had been asked where is f continuous,

the result would be $\boxed{(-\infty, -2) \cup (-2, 0) \cup (0, 2) \cup (2, \infty)}$

or you may write $\boxed{\mathbb{R} - \{0, -2, 2\}}$

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(14) $f(x) = \sec(x)$

Not continuous at the vertical asymptote values:

$\boxed{\text{Not continuous at ODD multiples of } \frac{\pi}{2}}$

OR $\boxed{\text{at } x = (2n+1)\left(\pi/2\right) \text{ for } n = 0, \pm 1, \pm 2, \ldots}$

(15) $f(x) = \begin{cases} \dfrac{8x}{x^2+1}, & x \le -1 \\ \dfrac{4}{x}, & x > -1 \end{cases}$

Possible Discontinuities:

(i) values where denominators equal zero

(ii) the change-of behavior value, $x=-1$

MUST CHECK THESE:

(i) TOP: $x^2+1=0 \Rightarrow x^2=-1 \Rightarrow x=\pm\sqrt{-1}$

NO SOLUTION

SO NO DISCONTINUITIES HERE

BOTTOM: $x=0 \Rightarrow$ DISCONTINUITY IS HERE.

(ii) $\lim_{x\to(-1)^-} f(x) \overset{TOP}{=} \lim_{x\to(-1)^-} \dfrac{8x}{x^2+1} = \dfrac{8(-1)}{(-1)^2+1} = -4$

$\lim_{x\to(-1)^+} f(x) \overset{BOT}{=} \lim_{x\to(-1)^+} \dfrac{4}{x} = \dfrac{4}{-1} = -4$

so $\lim_{x\to(-1)} = -4$

also $f(-1) \overset{TOP}{=} \dfrac{8(-1)}{(-1)+1} = -4$

also $\lim_{x\to(-1)} f(x) = f(-1) \Rightarrow$ SO NO DISCONTINUITY HERE.

THUS the only discontinuity is at $\boxed{x=0}$

(1b) $f(x) = \begin{cases} -\cos(x) & , \; x \le \pi \\ x & , \; x > \pi \end{cases}$

The only potential discontinuity is at $x = \pi$

must check: (i) $f(a)$ is defined

(ii) $\lim\limits_{x \to a} f(x)$ is defined

(iii) $\lim\limits_{x \to a} f(x) = f(a)$

$\Rightarrow$ check this for $a = \pi$

(i) $f(\pi) \overset{\text{TOP}}{=} -\cos(\pi) = -(-1) = 1$ ✓

(ii) $\lim\limits_{x \to \pi} f(x)$

(a) $\lim\limits_{x \to \pi^-} f(x) \overset{\text{TOP}}{=} \lim\limits_{x \to \pi^-} (-\cos(\pi)) = -\cos(\pi) = 1$ ✓

(b) $\lim\limits_{x \to \pi^+} f(x) \overset{\text{BOT}}{=} \lim\limits_{x \to \pi^+} (x) = \pi$ ✓

since those do not equal

$\lim\limits_{x \to \pi} f(x)$ D.N.E.

Thus $\boxed{f \text{ is DISCONTINUOUS AT } x = \pi}$

III

(17) $f(x) = \begin{cases} x^2+4x+2 \ , & x<1 \\ kx+3 \ , & x \geq 1 \end{cases}$

for f to be continuous we must have $\lim_{x\to 1} f(x)$ Defined

Thus, we must have:

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x)$$

$$\Rightarrow \underset{\text{TOP}}{\lim_{x\to 1^-} (x^2+4x+2)} = \underset{\text{BOTTOM}}{\lim_{x\to 1^+} (kx+3)}$$

$$\Rightarrow (1)^2+4(1)+2 = k(1)+3$$

$$\Rightarrow 7 = k+3$$

$$\Rightarrow \boxed{k=4}$$

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(18) $f(x) = \begin{cases} x^2-\sin(x)+k_1 \ ; & x<\pi \\ x^2-\cos(x)-2 \ ; & \pi \leq x < 2\pi \\ x^2-\tan(x)+k_2 \ ; & x \geq 2\pi \end{cases}$

Similarly to the above (#17) we have:

$$\lim_{x\to \pi^-} f(x) = \lim_{x\to \pi^+} f(x) \quad \underline{\text{AND}} \quad \lim_{x\to 2\pi^-} f(x) = \lim_{x\to 2\pi^+} f(x)$$

$$\Rightarrow \underset{\text{TOP}}{\lim_{x\to \pi^-} (x^2-\sin(x)+k_1)} = \underset{\text{MIDDLE}}{\lim_{x\to \pi^+} (x^2-\cos(x)-2)} \quad \underline{\text{AND}} \quad \overset{\text{MIDDLE}}{\lim_{x\to 2\pi^-} (x^2-\cos(x)-2)} = \underset{\text{BOTTOM}}{\lim_{x\to 2\pi^+} (x^2-\tan(x)+k_2)}$$

$$\Rightarrow (\pi)^2-\sin(\pi)+k_1 = (\pi)^2-\cos(\pi)-2 \quad \underline{\text{AND}} \quad (2\pi)^2-\cos(2\pi)-2 = (2\pi)^2-\tan(2\pi)+k_2$$

CONT.

CONT:

$\Rightarrow \pi^2 - 0 + k_1 = \pi^2 - (-1) - 2$ AND $4\pi^2 - 1 - 2 = 4\pi^2 - 0 + k_2$

$\Rightarrow \boxed{k_1 = -1}$ AND $\boxed{-3 = k_2}$

IIII.

(19) $\lim_{x \to 2} (4x-1) = 7$

Proof

I) Let $\epsilon > 0$

II) Search for δ:

we want $|f(x) - L| < \epsilon$

$\Rightarrow |(4x-1) - 7| < \epsilon$

$\Rightarrow |4x - 8| < \epsilon$

$\Rightarrow |4(x-2)| < \epsilon$

$\Rightarrow 4|x-2| < \epsilon$

$\Rightarrow |x-2| < \frac{\epsilon}{4}$

Choose $\delta = \frac{\epsilon}{4}$

Verify this δ:

$0 < |x-a| < \delta \Rightarrow$

$\Rightarrow |x-2| < \frac{\epsilon}{4}$

$\Rightarrow 4|x-2| < \epsilon$

$\Rightarrow |4(x-2)| < \epsilon$

$\Rightarrow |4x-8| < \epsilon$

$\Rightarrow |4x - 1 + 1 - 8| < \epsilon$

$\Rightarrow |(4x-1) - 7| < \epsilon$

$\Rightarrow |f(x) - L| < \epsilon$ ✓

(20) $\lim_{x \to (-2)} (-3x+4) = 10$

Proof:

I) Let $\epsilon > 0$

II) Search for δ:

we want $|f(x) - L| < \epsilon$

$\Rightarrow |(-3x+4) - 10| < \epsilon$

$\Rightarrow |-3x + 4 - 10| < \epsilon$

$\Rightarrow |-3x - 6| < \epsilon$

$\Rightarrow |-3(x+2)| < \epsilon$

$\Rightarrow |-3|\,|x - (-2)| < \epsilon$

$\Rightarrow |x - (-2)| < \frac{\epsilon}{|-3|}$

Choose $\delta = \frac{\epsilon}{|-3|}$

Verify this δ:

$0 < |x-a| < \delta \Rightarrow |x - (-2)| < \frac{\epsilon}{|-3|}$

$\Rightarrow |-3|\,|x+2| < \epsilon$

$\Rightarrow |-3(x+2)| < \epsilon$

$\Rightarrow |-3x-6| < \epsilon$

$\Rightarrow |-3x + 4 - 6 - 4| < \epsilon$

$\Rightarrow |(-3x+4) - 10| < \epsilon$

$\Rightarrow |f(x) - L| < \epsilon$ ✓