Solutions Exam 3 study

(I) ① $f(x) = x^4 + 4x^3 - 8x^2 - 10$ on $[-1, 2]$

$f'(x) = 4x^3 + 12x^2 - 16x$

Critical values

TYPE I $f' = 0$

TYPE II f' dne. None polynomial

$\Rightarrow 4x^3 + 12x^2 - 16x = 0$

$\Rightarrow 4x[x^2 + 3x - 4] = 0$

$\Rightarrow 4x(x+4)(x-1) = 0$

$x = 0$ $\cancel{x = -4}$ $x = 1$

Not in $[-1, 2]$

so

$f(0) = 0 + 0 - 0 - 10 = -10$

$f(1) = (1)^4 + 4(1)^3 - 8(1)^2 - 10$
$= 1 + 4 - 8 - 10 = -13$

$f(-1) = (-1)^4 + 4(-1)^3 - 8(-1)^2 - 10$
$= 1 - 4 - 8 - 10 = -21$ MIN

$f(2) = (2)^4 + 4(2)^3 - 8(2)^2 - 10$
$= 16 + 32 - 32 - 10 = 6$ MAX

$\boxed{\begin{array}{l} \text{ABS MAX} \Rightarrow (2, 6) \\ \text{ABS MIN} \Rightarrow (-1, -21) \end{array}}$

② $f(x) = \dfrac{x+2}{x^2+2x}$ on $[1, 4]$

$\Rightarrow f'(x) = \dfrac{(x^2+2x)(1) - (x+2)(2x+2)}{(x^2+2x)^2}$

$= \dfrac{x^2 + 2x - 2x^2 - 6x - 4}{(x^2+2x)^2}$

$= \dfrac{-x^2 - 4x - 4}{(x^2+2x)^2}$

$= \dfrac{(-1)(x^2 + 4x + 4)}{(x^2+2x)^2}$

Critical values:

TYPE I $f' = 0$

$\Rightarrow (-1)(x^2 + 4x + 4) = 0$

$\Rightarrow (x+2)(x+2) = 0$

$\Rightarrow \cancel{x = -2}$ NOT IN $[1, 4]$

TYPE II f' dne.

$\Rightarrow (x^2 + 2x)^2 = 0$

$\Rightarrow x^2 + 2x = 0$

$\Rightarrow x(x+2) = 0$

$\Rightarrow \cancel{x = 0}$ $\cancel{x = -2}$ NOT in $[1, 4]$

so $f(1) = \dfrac{(1)+2}{(1)^2 + 2(1)}$

$= \dfrac{3}{3} = 1$

$f(4) = \dfrac{(4)+2}{(4)^2 + 2(4)}$

$= \dfrac{6}{16+8}$

$= \dfrac{6}{24}$

$= \dfrac{1}{4}$

$\boxed{\begin{array}{l} \text{ABS MAX} \\ (1, 1) \\ \text{ABS MIN} \\ (4, 1/4) \end{array}}$

(I) (3) $f(x) = x^4 - 4x^3 + 4x^2 - 7$

$\Rightarrow f'(x) = 4x^3 - 12x^2 + 8x$

Critical values

Type I: $f' = 0$ Type II f' d.n.e. $\rightarrow$ None (polynomial)

$\Rightarrow 4x^3 - 12x^2 + 8x = 0$

$\Rightarrow 4x(x^2 - 3x + 2) = 0$

$\Rightarrow 4x(x-2)(x-1) = 0$

$x = 0 \quad x = 2 \quad x = 1$

$f''(x) = 12x^2 - 24x + 8$

$f''(0) = 12(0)^2 - 24(0) + 8 = 8 > 0$ Rel min

$f''(2) = 12(2)^2 - 24(2) + 8$
$= 48 - 48 + 8 = 8 > 0$ Rel min

$f''(1) = 12(1)^2 - 24(1) + 8$
$= 12 - 24 + 8 = -4 < 0$ Rel max

so $f(0) = (0) - (0) + (0) - 7 = -7$

$f(2) = (2)^4 - 4(2)^3 + 4(2)^2 - 7$
$= 16 - 32 + 16 - 7 = -7$

$f(1) = (1)^4 - 4(1)^3 + 4(1)^2 - 7$
$= 1 - 4 + 4 - 7 = -6$

so
Rel min at $(0, -7)$ & $(2, -7)$
Rel Max at $(1, -6)$

(4). $f(x) = x - 2\sin(x)$ on $(0, 2\pi)$

$\Rightarrow f'(x) = 1 - 2\cos(x)$

Critical values

TYPE I: $f' = 0$ TYPE II f' d.n.e NONE (cosine function)

$1 - 2\cos(x) = 0$

$\Rightarrow -2\cos(x) = -1$

$\Rightarrow \cos(x) = \frac{-1}{-2}$

$\Rightarrow \cos(x) = 1/2$

* Quadrant check (cosine positive)
QI QIIII

* Reference angle
$x_R = \pi/3$

* solve:
$x = \pi/3$
$x = 5\pi/3$

we have

$f''(x) = -2(-\sin(x))$
$= 2\sin(x)$

$f''(\pi/3) = 2\left(\frac{\sqrt{3}}{2}\right) = \sqrt{3} \approx 1.7 > 0$ Rel min.

$f''(5\pi/3) = 2\left(-\frac{\sqrt{3}}{2}\right) = -\sqrt{3} \approx -1.6 < 0$ Rel max

so $f(\pi/3) = (\pi/3) - 2\sin(\pi/3)$
$= \pi/3 - 2\frac{\sqrt{3}}{2}$
$= \pi/3 - \sqrt{3}$

$f(5\pi/3) = \frac{5\pi}{3} - 2\sin(5\pi/3)$
$= 5\pi/3 - 2(-\sqrt{3}/2)$
$= 5\pi/3 + \sqrt{3}$

so we have:
Rel max at $(5\pi/3, 5\pi/3 + \sqrt{3})$
Rel min at $(\pi/3, \pi/3 - \sqrt{3})$

III. (5) $f(x) = (x^2-4)^{2/3}$

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PERFORM a 1st derivative analysis.

$f'(x) = \frac{2}{3}(x^2-4)^{-1/3}(2x)$

$= \dfrac{2 \cdot 2x}{(x^2-4)^{1/3}} = \dfrac{4x}{(x^2-4)^{1/3}}$

Critical values:

TYPE I $f'=0$
set num $=0$
$\Rightarrow 4x = 0$
$\Rightarrow x = 0$

TYPE II f' d.n.e.
set denom $=0$
$(x^2-4)^{1/3} = 0$
$\Rightarrow x^2-4 = 0$
$\Rightarrow x^2 = 4$
$\Rightarrow x = \pm 2$

$\searrow$ $\nearrow$ $\searrow$ $\nearrow$

$\boxed{-2}$ $\boxed{0}$ $\boxed{2}$

$f'(-3) = \dfrac{4(-3)}{((-3)^2-4)^{1/3}} = \dfrac{-12}{5^{1/3}}$ $\dfrac{\text{NEG}}{\text{POS}} \Rightarrow \text{NEG} < 0$ $\searrow$

$f'(-1) = \dfrac{4(-1)}{((-1)^2-4)^{1/3}} = \dfrac{-4}{(-3)^{1/3}}$ $\dfrac{\text{NEG}}{\text{NEG}} \Rightarrow \text{POS} > 0$ $\nearrow$

$f'(1) = \dfrac{4(1)}{((1)^2-4)^{1/3}} = \dfrac{4}{(-3)^{1/3}}$ $\dfrac{\text{POS}}{\text{NEG}} \Rightarrow \text{NEG} < 0$ $\searrow$

$f'(3) = \dfrac{4(3)}{((3)^2-4)^{1/3}} = \dfrac{12}{(5)^{1/3}}$ $\dfrac{\text{POS}}{\text{POS}} \Rightarrow \text{POS} > 0$ $\nearrow$

so

$f \searrow (-\infty, -2)$ and $(0, 2)$

$f \nearrow (-2, 0)$ and $(2, \infty)$

$f(-2) = ((-2)^2-4)^{2/3} = 0$

$f(0) = ((0)^2-4)^{2/3} = (-4)^{2/3}$
$= ((-4)^2)^{1/3}$
$= (16)^{1/3}$

$f(2) = ((2)^2-4)^{2/3} = 0$

at $x=-2$ changes from $\searrow$ to $\nearrow$

at $x=0$ changes from $\nearrow$ to $\searrow$

at $x=2$ changes from $\searrow$ to $\nearrow$

so

Rel Min at $(-2, 0)$ and at $(2, 0)$

Rel Max at $(0, 16^{1/3})$

⑥. $f(x) = x + \cos(x)$ on $(0, \pi)$ (1st derivative analysis)

$f'(x) = 1 - \sin(x)$

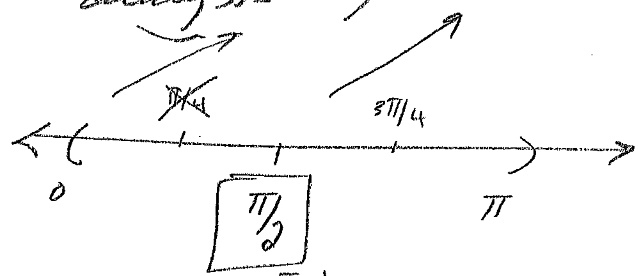
Critical Values

TYPE I $f' = 0$ TYPE II f' d.n.e none ($\sin(x)$)

$1 - \sin(x) = 0$

$\Rightarrow \sin(x) = 1$ (Quadrantal)

$\Rightarrow x = \pi/2$



$f'(\pi/4) = 1 - \sin(\pi/4)$

$= 1 - \sqrt{2}/2$

$\approx 1 - .707$

$\approx .3 > 0 \nearrow$

$f'(3\pi/4) = 1 - \sin(3\pi/4)$

$= 1 - \frac{\sqrt{2}}{2}$ $\approx 1 - .707$

$\approx .3 > 0 \nearrow$

so

$f \nearrow$ on $(0, \pi/2)$ and $(\pi/2, \pi)$

since there is no change at $x = \pi/2$ it is neither max nor min.

III ⑦ $f(x) = \dfrac{x^2+1}{x^2-4}$ $\quad f'(x) = \dfrac{-10x}{(x^2-4)^2}$ $\quad f''(x) = \dfrac{30x^2+40}{(x^2-4)^3}$

Ⓐ Critical Values:

TYPE I $f' = 0$ set num $= 0$

$\Rightarrow -10x = 0$

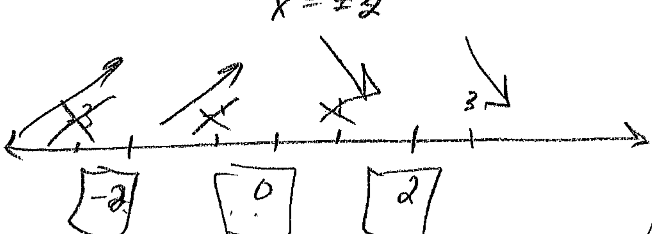
$\Rightarrow x = 0$

TYPE II f' d.n.e. set denom $= 0$

$(x^2-4)^2 = 0$

$x^2 - 4 = 0$

$x = \pm 2$



$f'(-3) = \dfrac{(-10)(-3)}{((-3)^2-4)^2} = \dfrac{pos}{pos} > 0 \nearrow$

$f'(-1) = \dfrac{(-10)(-1)}{((-1)^2-4)^2} = \dfrac{pos}{pos} > 0 \nearrow$

$f'(1) = \dfrac{(-10)(1)}{((1)^2-4)^2} = \dfrac{NEG}{pos} < 0 \searrow$

$f'(3) = \dfrac{(-10)(3)}{((3)^2-4)^2} = \dfrac{NEG}{pos} < 0 \searrow$

so $f \nearrow$ on $(-\infty, -2)$ and $(-2, 0)$

$f \searrow$ on $(0, 2)$ and $(2, \infty)$

#7 continued

#7 cont

(B) potential max at $x=0$ (since changes from ↗ to ↘)

$$f(0) = \frac{(0)^2+1}{(0)^2-4} = -\frac{1}{4} \Rightarrow \text{so } \boxed{\text{Rel max at } (0, -1/4)}$$

↳ Note: if this had resulted in "d.n.e." then there is no rel max at $x=0$

(C) Conduct a 2nd derivative analysis:

① Critical Values:

TYPE I: $f''=0$, num $=0$

$\Rightarrow 30x^2+40=0$

$\Rightarrow 30x^2=-40$

$\Rightarrow x^2 = -40/30$

$\Rightarrow x = \pm\sqrt{-40/30}$

No soln

so NO TYPE I

TYPE II: f'' d.n.e., denom $=0$

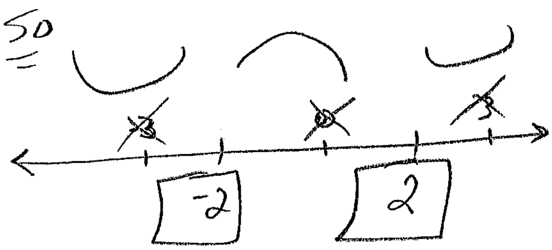
$(x^2-4)^3=0$

$\Rightarrow x^2-4=0$

$\Rightarrow x^2=4$

$\Rightarrow x=\pm 2$

so



$$f''(-3) = \frac{30(-3)^2+40}{((-3)^2-4)^3} = \frac{\text{pos}}{\text{pos}} > 0 \ \cup$$

$$f''(0) = \frac{30(0)^2+40}{((0)^2-4)^3} = \frac{\text{pos}}{\text{NEG}} < 0 \ \cap$$

$$f''(3) = \frac{30(3)^2+40}{((3)^2-4)^3} = \frac{\text{pos}}{\text{pos}} > 0 \ \cup$$

so we have:

f is $\cup$ on $(-\infty, -2)$ and $(2, \infty)$

f is $\cap$ on $(-2, 2)$

(D) potential inflection points at $x=-2$ and $x=2$

$$f(-2) = \frac{(-2)^2+1}{(-2)^2-4} = \frac{5}{0} \text{ D.N.E.}$$

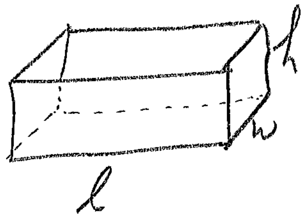
NO INFLECTION PT AT $x=-2$

$$f(2) = \frac{(2)^2+1}{(2)^2-4} = \frac{5}{0} \text{ D.N.E.}$$

NO INFLECTION PT at $x=2$

so $\boxed{\text{There are NO inflection points}}$

#8



Find $l =$ cm
$w =$ cm
$h =$ cm

Assume:

$l = 3 \cdot w$

$l + w + h = 120$

$\Rightarrow$ so $\underbrace{l}_{3 \cdot w} + w + h = 120$

$\Rightarrow 3w + w + h = 120$

$\Rightarrow 4w + h = 120$

$\Rightarrow h = 120 - 4w$

relationship statement

NOW

We wish to maximize Volume so we need Volume as a function of a single variable:

$V = l \cdot w \cdot h$

$= (3w)(w)(h)$

must eliminate a variable (w or h)

I eliminated h

$V = (3w)(w)(120 - 4w)$

so we have

$V(w) = (3w)(w)(120 - 4w)$

(you could use x, but I went with "w")

$\Rightarrow V(w) = 3w^2(120 - 4w)$

$\Rightarrow V(w) = 360w^2 - 12w^3$

this is my volume function

Need the interval of definition:

smallest (theoretical) value for w is 0 and largest is found from the relationship statement with $h = 0$

$4w + h = 120$

$\Rightarrow 4w + (0) = 120$

$\Rightarrow w = 30$

so we have

$V(w) = 360w^2 - 12w^3$

$[0, 30]$

#8 cont

#8 cont

$V(w) = 360w^2 - 12w^3$ on $[0, 30]$

$\Rightarrow V'(w) = 720w - 36w^2$

critical values

TYPE I $V' = 0$ — TYPE II V' d.n.e. NONE (polynomial)

$720w - 36w^2 = 0$

$\Rightarrow 36w(20 - w) = 0$

$\Rightarrow w = 0$ or $w = 20$

$V(0) = 360(0)^2 - 12(0)^3 = 0 - 0 = 0$

$V(20) = 360(20)^2 - 12(20)^3 = 144000 - 96000 = 48000$

$V(30) = 360(30)^2 - 12(30)^3 = 324000 - 324000 = 0$

so $w = 20$

$l = 3(w) = 60$

$h = 120 - 4(20) = 120 - 80 = 40$

$$\boxed{\begin{array}{l} l = 20\text{ cm} \\ w = 60\text{ cm} \\ w = 40\text{ cm} \end{array}}$$

#9. MAXIMIZE AREA

Find: "dimensions of each shed" $\Rightarrow$ $\boxed{\begin{array}{l} x = \quad m \\ y = \quad m \end{array}}$

$A = (2x)(4y)$

$\Rightarrow A = 8xy$

must eliminate a variable

we have "total fence" $= 120$m

$12x + 4y = 120 \Leftarrow$ relationship statement

$\Rightarrow 4y = -12x + 120$

$\Rightarrow y = -3x + 30$

so $A(x) = 8x(-3x + 30)$

$\Rightarrow A(x) = -24x^2 + 240x$

$\Rightarrow A(x) = -24x^2 + 240x$

Need interval

minimum value for $x = 0$

Maximum value for x

$\Rightarrow$ go to relationship statement & set $y = 0$

$\Rightarrow 12x + 0 = 120$

$\Rightarrow x = \frac{120}{12} = 10$

so MAXIMIZE

$A(x) = -24x^2 + 240x$ on $[0, 10]$

cont

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$\Rightarrow$ $A(x) = -24x^2 + 240x$
on $[0, 10]$

$\Rightarrow$ $A'(x) = -48x + 240$
critical values

TYPE I $A' = 0$ TYPE II A' D.N.E.

$\Rightarrow$ $-48x + 240 = 0$

$\Rightarrow$ $-48x = -240$

$\Rightarrow$ $x = \frac{-240}{-48} = 5$

so from the relationship statement:

$y = -3(5) + 30$
$= 15$

so $\boxed{\begin{array}{l} x = 5\,m \\ y = 15\,m \end{array}}$