Solutions

EXAM I

I

1. E
2. A
3. D
4. B
5. A
6. C
7. C
8. D
9. C
10. A
11. E
12. E
13. E
14. A

II

20. $f(x) = \begin{cases} \dfrac{-3}{x^2-4} & \text{if } x \le 1 \\ 3-x^2 & \text{if } x > 1 \end{cases}$

Potential Discontinuity if

$x^2-4=0$

$\Rightarrow x = \pm 2$

But only $x=-2$ is in the interval $x \le 1$ so there is a discontinuity at $x=-2$

We must investigate the "change of behavior value" at $x=1$

Use the "continuity checklist:"

(a) $f(1) \overset{TOP}{=} \dfrac{-3}{(1)^2-4} = \dfrac{-3}{-3} = 1$

so $f(1)$ exists ✓

(b) $\lim\limits_{x\to 1} f(x)$ must check the one-sided limits

(i) $\lim\limits_{x\to 1^-} f(x) \overset{TOP}{=} \lim\limits_{x\to 1^-} \left(\dfrac{-3}{(1)^2-4}\right) = 1$ ✓

(ii) $\lim\limits_{x\to 1^+} f(x) \overset{BOT}{=} \lim\limits_{x\to 1^+} (3-x^2) = 3-(1)^2 = 2$ ✓

But these do not match

so $\lim\limits_{x\to 1} f(x)$ D.N.E

so Discontinuity at $x=1$

Thus $f(x)$ is NOT CONTINUOUS at $\boxed{\begin{matrix} x=-2 \\ x=1 \end{matrix}}$

(21) $f(x) = \begin{cases} x^2 - kx & \text{if } x < -2 \\ 4x+2 & \text{if } x \geq -2 \end{cases}$

To be continuous we must have

$$\lim_{x \to (-2)^-} f(x) = \lim_{x \to (-2)^+} f(x)$$

$$\Rightarrow \overset{\text{TOP}}{\lim_{x \to (-2)^-}} (x^2 - kx) = \overset{\text{BOT}}{\lim_{x \to (-2)^+}} (4x+2)$$

$$\Rightarrow (-2)^2 - k(-2) = 4(-2) + 2$$

$$\Rightarrow 4 + 2k = -6$$

$$\Rightarrow 2k = -10$$

$$\Rightarrow k = -5$$

$\boxed{\text{so } f(x) \text{ is continuous if } k = -5}$

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(22). $\lim_{x \to 5} \frac{(\sqrt{x-1} - 2)}{(x-5)}$ $\Rightarrow$ NOTICE DIRECT SUBSTITUTION GIVES $\frac{\sqrt{5-1} - 2}{5-5} \Rightarrow \frac{0}{0}$ so:

"multiply by the conjugate"

we have $\lim_{x \to 5} \frac{(\sqrt{x-1} - 2)}{(x-5)} \cdot \frac{(\sqrt{x-1} + 2)}{(\sqrt{x-1} + 2)} =$

$$= \lim_{x \to 5} \frac{(\sqrt{x-1})^2 - (2)^2}{(x-5)(\sqrt{x-1} + 2)}$$

$$= \lim_{x \to 5} \frac{x - 1 - 4}{(x-5)(\sqrt{x-1} + 2)}$$

$$= \lim_{x \to 5} \frac{\cancel{(x-5)}}{\cancel{(x-5)}(\sqrt{x-1} + 2)}$$

$$= \lim_{x \to 5} \frac{1}{(\sqrt{x-1} + 2)}$$

$$= \frac{1}{(\sqrt{5-1} + 2)}$$

$$= \frac{1}{2+2}$$

$$= \boxed{\frac{1}{4}}$$

(23). $f(x) = \begin{cases} -\sin(x) & \text{if } x \le 0 \\ \tan(x) & \text{if } x > 0 \end{cases}$ on the interval $(-2\pi, 2\pi)$

If $x<0$ $f(x) = -\sin(x)$ which has NO Discontinuities

If $x>0$ $f(x) = \tan(x)$ which has discontinuities at $x = \pi/2$ and $x = 3\pi/2$

Recall the restriction $(-2\pi, 2\pi)$

If $x=0$ must use the 3-point continuity definition:

ⓐ $f(0) \overset{TOP}{=} -\sin(0) = 0$

ⓑ $\lim_{x\to 0} f(x)$

ⓘ $\lim_{x\to 0^-} f(x) \overset{TOP}{=} \lim_{x\to 0^-} (-\sin(x)) = -\sin(0) = 0$

ⓘⓘ $\lim_{x\to 0^+} f(x) \overset{BOT}{=} \lim_{x\to 0^+} (\tan(x)) = \tan(0) = 0$

Thus $\lim_{x\to 0} f(x) = 0$

ⓒ $f(0) = \lim_{x\to 0} f(x)$

Thus continuous at $x=0$

Thus the only discontinuities are $x = \pi/2$ and $x = 3\pi/2$

(24). $\lim_{x \to 5} \dfrac{x^2 - 7x + 10}{x^2 - 2x + 15}$ NOTE: Direct substition gives:

$$\frac{(5)^2 - 7(5) + 10}{(5)^2 - 2(5) + 15} \begin{matrix} \to 0 \\ \to 0 \end{matrix}$$

So $\lim_{x \to 5} \dfrac{x^2 - 7x + 10}{x^2 - 2x + 15} =$

$$= \lim_{x \to 5} \frac{(x-2)\cancel{(x-5)}}{\cancel{(x-5)}(x+3)}$$

$$= \lim_{x \to 5} \frac{(x-2)}{(x+3)}$$

$$= \frac{(5)-2}{(5)+3}$$

$$= \boxed{\frac{3}{8}}$$

(25) $\lim_{x \to 0} \dfrac{5\tan(3x)}{4x}$ NOTE: Direct substitution gives $\dfrac{5\tan(3(0))}{4(0)} \begin{matrix} \to 0 \\ \to 0 \end{matrix}$

So: $\lim_{x \to 0} \dfrac{5\tan(3x)}{4x} = \lim_{x \to 0} \dfrac{5\sin(3x)}{\cos(3x) \cdot 4x}$

$$= \lim_{x \to 0} \frac{5}{4\cos(3x)} \cdot \frac{\sin(3x)}{x}$$

must make these match

$$= \lim_{x \to 0} \frac{5 \cdot 3 \cdot \sin(3x)}{4\cos(3x) \cdot 3 \cdot x}$$

$$= \lim_{x \to 0} \frac{5 \cdot 3}{4 \cdot \cos(3x)} \cdot \lim_{x \to 0} \frac{\sin(3x)}{3x}$$

$$= \frac{15}{4(\cos(3(0))} \cdot 1$$

$$= \boxed{\frac{15}{4}}$$

#26. $f(x) = \begin{cases} 4x - x^2 & \text{if } x \le -1 \\ \dfrac{15}{x+4} & \text{if } x > -1 \end{cases}$

(a) $f(-1) \overset{TOP}{=} 4(-1) - (-1)^2 = -4 - 1 = \boxed{-5}$

so $f(-1)$ exists

(b) $\lim\limits_{x \to (-1)} f(x)$

must check one-sided limits

(i) $\lim\limits_{x \to (-1)^-} f(x) \overset{TOP}{=} \lim\limits_{x \to (-1)^-} (4x - x^2)$

$= 4(-1) - (-1)^2$

$= -4 - 1$

$= \boxed{-5}$

(ii) $\lim\limits_{x \to (-1)^+} f(x) \overset{BOT}{=} \lim\limits_{x \to (-1)^+} \left(\dfrac{15}{x+4}\right)$

$= \dfrac{15}{(-1)+4}$

$= \dfrac{15}{3}$

$= \boxed{5}$

But these do not agree

so $\lim\limits_{x \to (-1)} f(x)$ D.N.E.

so $\boxed{f(x) \text{ is NOT CONTINUOUS at } x = -1}$

(27) Prove $\lim_{x \to (-4)} (-3x-6) = 6$

(I) Let $\epsilon > 0$

(II) Search for δ

we want

$|f(x) - L| < \epsilon \Rightarrow$

$\Rightarrow |(-3x-6) - 6| < \epsilon$

$\Rightarrow |-3x-12| < \epsilon$

$\Rightarrow |-3(x+4)| < \epsilon$

$\Rightarrow |-3|\,|x-(-4)| < \epsilon$

$\Rightarrow |x-(-4)| < \frac{\epsilon}{|-3|}$

choose

$\delta = \frac{\epsilon}{|-3|}$

(III) Verify this δ:

$0 < |x-a| < \delta \Rightarrow$

$\Rightarrow |x-(-4)| < \frac{\epsilon}{|-3|}$

$\Rightarrow |-3|\,|x+4| < \epsilon$

$\Rightarrow |-3(x+4)| < \epsilon$

$\Rightarrow |-3x-12| < \epsilon$

$\Rightarrow |-3x-6+6-12| < \epsilon$

$\Rightarrow |(-3x-6)-6| < \epsilon$

$\Rightarrow |f(x) - L| < \epsilon$