Solutions

pg 1

I ① $\sum_{n=0}^{\infty} 8(-.25)^n \Rightarrow \sum_{n=0}^{\infty} 8(-.25)^n$

Note: there is a typo right off the bat: this sum should start with $n=0$ and have n here

Geometric $a=8$ $r=-.25$

$|r| = |-.25| = .25 < 1$

$\therefore$ $\boxed{\text{converges}}$

the sum is:

$S = \frac{a}{1-r} = \frac{8}{1-.25} = \frac{8}{.75} = \boxed{\frac{32}{3}}$

② $\sum_{n=1}^{\infty} \frac{n-2}{n^5-n^3-1}$ $\Rightarrow$ Fraction of polynomials so use Limit comparison test.

Limit comparison Test

vs: "fraction of dominant terms"

test vs $\sum_{n=1}^{\infty} \frac{n}{n^5} = \sum_{n=1}^{\infty} \frac{1}{n^4} \Rightarrow$ p-series $p=4>1 \Rightarrow$ converges

$$\lim_{n\to\infty} \frac{\frac{n-2}{n^5-n^3-1}}{\frac{1}{n^4}} = \lim_{n\to\infty} \frac{(n-2)}{(n^5-n^3-1)} \cdot \frac{n^4}{1}$$

$$= \lim_{n\to\infty} \frac{n^5-2n^4}{n^5-n^3-1}$$ ← divide them by largest power of n in the denominator.

$$= \lim_{n\to\infty} \frac{n^5/n^5 - 2n^4/n^5}{n^5/n^5 - n^3/n^5 - 1/n^5}$$

$$= \lim_{n\to\infty} \frac{1 - \cancel{2/n}}{1 - \cancel{1/n^2} - \cancel{1/n^5}} \to 0$$

$= \frac{1}{1}$

$= 1 \Rightarrow$ finite & positive $\therefore$ they agree

$\Rightarrow$ $\boxed{\text{the series converges}}$

③ $\sum_{n=1}^{\infty}\left(\frac{2}{2n-3}-\frac{2}{2n-1}\right)$ consider the leading term $\frac{2}{2n-3} \Rightarrow \frac{2}{2(n+1)-3} = \frac{2}{2n+2-3}$ Pg two

$= \frac{2}{2n-1} \Rightarrow$ the following term

Telescoping series:

$\lim_{n\to\infty} \frac{2}{2n-1} = \frac{2}{\infty} \to 0$ which is finite

$\therefore$ It is telescoping

$\therefore$ the series converges

The sum is: $S = \frac{2}{2n-3}\Big|_{n=1} - 0$

$= \frac{2}{2(1)-3} - 0$

$= \frac{2}{-1} = \boxed{-2}$

④. $\sum_{n=1}^{\infty} \frac{(-1)^n(n-3)}{(n+1)!}$ → this involves a factorial ⇒ Ratio test.

Ratio Test: $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+1}((n+1)-3)}{((n+1)+1)!}}{\frac{(-1)^n(n-3)}{(n+1)!}}\right|$

$= \lim_{n\to\infty}\left|\frac{(-1)^{n+1}(n-2)}{(n+2)!}\cdot\frac{(n+1)!}{(-1)^n(n-3)}\right|$

$= \lim_{n\to\infty} \frac{(n-2)(n+1)!}{(n+2)!(n-3)}$

$= \lim_{n\to\infty} \frac{(n-2)\cancel{(n+1)!}}{(n+2)\cancel{(n+1)!}(n-3)}$

$= \lim_{n\to\infty} \frac{(n-2)}{(n+2)(n-3)}$

$= \lim_{n\to\infty} \frac{(n-2)}{n^2-n-6}$

$\overset{L'H}{=} \lim_{n\to\infty} \frac{1}{2n-1} \to 0$

$= 0 < 1$

$\boxed{\therefore \text{the series converges}}$

⑤ $\sum_{n=0}^{\infty} \frac{(-1)^{n+1}(2\pi)^n}{(2n)!}$ ⇒ this involves a factorial so the Ratio Test is in order. pg Shree

Ratio Test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(-1)^{(n+1)+1}(2\pi)^{n+1}}{(2(n+1))!}}{\frac{(-1)^{n+1}(2\pi)^n}{(2n)!}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{\cancel{(-1)^{n+2}}(2\pi)^{n+1}}{(2n+2)!}\cdot\frac{(2n)!}{\cancel{(-1)^{n+1}}(2\pi)^n}\right|$$

$$= \lim_{n\to\infty}\frac{(2\pi)^1\,\cancel{(2n)!}}{(2n+2)(2n+1)\cancel{(2n)!}}$$

$$= \lim_{n\to\infty}\frac{2\pi}{4n^2+6n+2}$$

$$= \frac{2\pi}{\infty}$$

$$= 0 < 1 \Rightarrow \boxed{\text{the series converges}}$$

⑥ $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}[3\cdot5\cdot7\cdot\ldots\cdot(2n+1)]}{(2n)!}$ ⇒ involves "factorial type" and factorial ⇒ Ratio Test.

Ratio test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(-1)^{(n+1)+1}[3\cdot5\cdot7\ldots(2n+1)(2(n+1)+1)]}{(2(n+1))!}}{\frac{(-1)^{n+1}[3\cdot5\cdot7\cdot\ldots\cdot(2n+1)]}{(2n)!}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{\cancel{(-1)^{n+2}}[\cancel{3\cdot5\cdot7\cdot\ldots\cdot(2n+1)}(2n+3)]}{(2n+2)!}\cdot\frac{(2n)!}{\cancel{(-1)^{n+1}}[\cancel{3\cdot5\cdot7\cdot\ldots\cdot(2n+1)}]}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(2n+3)\cancel{(2n)!}}{(2n+2)(2n+1)\cancel{(2n)!}}\right|$$

$$= \lim_{n\to\infty}\frac{2n+3}{4n^2+6n+2} \overset{L'H}{=} \lim_{n\to\infty}\frac{2}{8n+6}^{\nearrow 0} = 0 < 1 \;\boxed{\therefore \text{the series converges}}$$

The second batch of problems starts here. pg four

Note there is a typo for Item #2. It should read $\sum_{n=6}^{\infty}$

① $\sum_{n=1}^{\infty}\left(\frac{n^2}{n^2-3}-\frac{(n+1)^2}{n^2+2n-2}\right) \Rightarrow$ check to see if this is telescoping:

Initial term $\Rightarrow \frac{n^2}{n^2-3} \Rightarrow \frac{(n+1)^2}{(n+1)^2-3} \Rightarrow \frac{(n+1)^2}{(n^2+2n+1-3} \Rightarrow \frac{(n+1)^2}{(n^2+2n-2)}$

this is the second term

$\therefore$ this is telescoping:

we have: $\lim_{n\to\infty} \frac{(n+1)^2}{n^2+2n-2} = \lim_{n\to\infty} \frac{n^2+2n+1}{n^2+2n-2} \stackrel{L'H}{=} \lim_{n\to\infty} \frac{2n+2}{2n+2} = \lim_{n\to\infty} 1 = 1.$

This is finite

Thus it is telescoping.

The sum is: $S = \left.\frac{n^2}{n^2-3}\right|_{n=1} - 1$

$= \frac{1}{1-3} - 1$

$= -\frac{1}{2} - 1 = \boxed{-\frac{3}{2}}.$

② $\sum_{n=1}^{\infty} \frac{n^2-2}{n^4+4} \Rightarrow$ Fraction of polynomials $\Rightarrow$ Limit comparison test

Limit Comparison Test:

Test vs "Fraction of Dominant terms. $\rightarrow \sum_{n=1}^{\infty}\frac{n^2}{n^4} = \sum_{n=1}^{\infty}\frac{1}{n^2} \Rightarrow$ p-series $p=2>1 \Rightarrow$ converges

$\lim_{n\to\infty} \frac{\frac{n^2-2}{n^4+4}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{(n^2-2)}{n^4+4}\cdot\frac{n^2}{1}$

$= \lim_{n\to\infty} \frac{n^4-2n^2}{n^4+4}$

$= \lim_{n\to\infty} \frac{n^4/n^4 - 2n^2/n^4}{n^4/n^4 + 4/n^4}$

$= \lim_{n\to\infty} \frac{1-\cancel{2/n^2}^{\to 0}}{1+\cancel{4/n^4}^{\to 0}}$

$= \lim_{n\to\infty} (1)$

$= 1 \Rightarrow$ finite & positive $\therefore$ They agree

$\boxed{\therefore \text{ the series converges}}$

③. $\sum_{n=0}^{\infty} 5(\pi-3)^n \Rightarrow$ Geometric

Pg 5 ive

with $a = 5$ $r = (\pi - 3)$

$|r| = |\pi-3| < 1 \Rightarrow$ Converges

The series $\boxed{\text{Converges}}$

$\Rightarrow$ The sum is: $S = \frac{a}{1-r}$

$= \frac{5}{1-(\pi-3)} = \frac{5}{1-\pi+3} = \boxed{\frac{5}{4-\pi}}$

④. $\sum_{n=1}^{\infty} \frac{(-1)^n e^{2n}}{(n+1)!}$ $\Rightarrow$ involves exponential and factorial so use the Ratio Test.

Ratio Test:

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+1} e^{(2(n+1))}}{((n+1)+1)!}}{\frac{(-1)^n e^{2n}}{(n+1)!}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(-1)^{n+1} e^{(2n+2)}}{(n+2)!} \cdot \frac{(n+1)!}{(-1)^n (e^{2n})} \right|$$

$$= \lim_{n\to\infty} \frac{e^{2n+2-2n} \cancel{(n+1)!}}{(n+2)\cancel{(n+1)!}}$$

$$= \lim_{n\to\infty} \frac{e^2}{n+2} \to 0$$

$= 0 < 1 \Rightarrow$ $\boxed{\text{the series converges}}$

5. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}[2\cdot 5\cdot 8 \cdot \ldots \cdot (3n-1)]}{n!}$ → involves "factorial like" and factorial ⇒ Ratio Test

PJ loix

Ratio Test:

$$\lim \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+2}[2\cdot 5\cdot 8 \cdot \ldots \cdot (3n-1)(3(n+1)-1)]}{(n+1)!}}{\frac{(-1)^{n+1}[2\cdot 5\cdot 8\cdot \ldots \cdot (3n-1)]}{n!}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{\cancel{(-1)^{n+2}}[\cancel{2}\cdot\cancel{5}\cdot\cancel{8} \cdot \ldots \cdot (3n\cancel{-1})(3n+2)]}{(n+1)!} \cdot \frac{n!}{\cancel{(-1)^{n+1}}[\cancel{2}\cdot\cancel{5}\cdot\cancel{8} \cdot \ldots \cdot (3n\cancel{-1})]} \right|$$

$$= \lim_{n\to\infty} \frac{(3n+2)(\cancel{n!})}{(n+1)(\cancel{n!})}$$

$$= \lim_{n\to\infty} \frac{3n+2}{n+1}$$

$$\overset{L'H}{=} \lim_{n\to\infty} \frac{3}{1} = 3 > 1 \therefore \boxed{\text{Diverges}}$$

6. $\sum_{n=0}^{\infty} \frac{(-1)^n (x-2)^n}{(2n)!}$

Ratio test:

$$\lim_{n\to\infty} \left| \frac{\cancel{(-1)^{n+1}} (x-2)^{n+1}}{(2(n+1))!} \cdot \frac{(2n)!}{\cancel{(-1)^n} (x-2)^n} \right| =$$

$$= \lim_{n\to\infty} \left| \frac{(x-2)^{n+1} (2n)!}{(2n+2)! \, (x-2)^n} \right|$$

$$= \lim_{n\to\infty} \frac{|x-2| \cancel{(2n)!}}{(2n+2)(2n+1)\cancel{(2n)!}}$$

$$= |x-2| \lim_{n\to\infty} \frac{1}{(2n+2)(2n+1)} \to 0$$

$= 0 < 1$ for all values of x

$\therefore$ Converges on $\boxed{(-\infty, \infty)}$

7. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x+1)^n}{(n+5)}$

Ratio Test:

$$\lim_{n\to\infty} \left| \frac{(-1)^{n+2}(x+1)^{n+1}}{(n+6)} \cdot \frac{(n+5)}{(-1)^{n+1}(x+1)^n} \right| =$$

$$= \lim_{n\to\infty} \left| \frac{(x+1)(n+5)}{(n+6)} \right|$$

$$= |x+1| \cdot \lim_{n\to\infty} \frac{n+5}{n+6}$$

$$\overset{L'H}{=} |x+1| \cdot \lim_{n\to\infty} \frac{1}{1}$$

$$= |x+1|$$

converges if $|x+1| < 1$

$\Rightarrow -1 < x+1 < 1$

$\Rightarrow -2 < x < 0$

must check end pts:

① $x=-2 \Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(-1)^n}{n+5} = \sum \frac{(-1)^{2n+1}}{n+5}$

$= \sum \frac{-1}{n+5}$

$= -1 \sum \frac{1}{n+5} \Rightarrow$ diverges

② $x=0 \Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(1)^n}{n+5} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n+5}$ converges

$\therefore$ The interval of convergence is: $\boxed{(-2, 0]}$

---

8. $\int_0^1 \cos(x^{1/2})\,dx$

$$\cos(x^{1/2}) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(x^{1/2})^{2n}}{(2n)!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^n}{(2n)!}$$

So we have:

$$\int_0^1 \sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^n}{(2n)!}\,dx =$$

$$= \left[\sum_{n=0}^{\infty} \frac{(-1)^{n+1} n x^{n-1}}{(2n)!} \right]_0^1$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^{n+1} n (1)^{n-1}}{(2n)!} - \sum_{n=0}^{\infty} \frac{(-1)^{n+1} n (0)^{n-1}}{(2n)!}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^{n+1} n}{(2n)!} \approx \frac{(-1)^{0+1}(0)}{(2(0))!} + \frac{(-1)^{1+1}(1)}{(2(1))!} + \frac{(-1)^{2+1}(2)}{2(2)!}$$

$$= 0 + \frac{1}{2} - \frac{2}{4} = \boxed{1/4}$$