Solutions

(I)

1. $\sum_{n=0}^{\infty} 16(-.3)^n$

Geometric
$a=16 \quad r=-.3$
$|r| = |-.3| = .3 < 1$
$\therefore$ converges

$\Rightarrow$ the sum is:

$$S = \frac{a}{1-r} = \frac{16}{1-(-.3)} = \boxed{\frac{16}{1.3}} \approx \boxed{12.308}$$

good enough

(*) also do these: 1b $\sum_{n=3}^{\infty} 12\left(-\frac{2}{3}\right)^n$ 1c $\sum_{n=0}^{\infty} 6\left(\frac{7}{3}\right)^n$

solutions to this at the end of solutions.

2. $\sum_{n=1}^{\infty} \frac{n+4}{n^6+n^4-1}$

Limit comparison test (fraction of algebraic expressions)

Compare with "ratio of dominant terms"

$\Rightarrow$ Compare with $\sum_{n=1}^{\infty} \frac{n}{n^6} = \sum_{n=1}^{\infty} \frac{1}{n^5}$

This is a p-series with $p=5$ $\therefore \sum_{n=1}^{\infty} \frac{1}{n^5}$ converges.

we have:

$$\lim_{n\to\infty} \frac{\frac{n+4}{n^6+n^4-1}}{\frac{1}{n^5}} = \lim_{n\to\infty} \frac{n+4}{n^6+n^4-1} \cdot \frac{n^5}{1}$$

$$= \lim_{n\to\infty} \frac{n^6+4n^5}{n^6+n^4-1}$$

$$= \lim_{n\to\infty} \frac{n^6/n^6 + 4n^5/n^6}{n^6/n^6 + n^4/n^6 - 1/n^6}$$

$$= \lim_{n\to\infty} \frac{1+\cancel{4/n}^{\,0}}{1+\cancel{1/n^2}-\cancel{1/n^6}^{\,0}}$$

$= 1$ this is finite and positive

$\therefore$ the series agree $\Rightarrow$ $\therefore$ The given series converges.

③. $\sum_{n=1}^{\infty} \left(\frac{2}{2n-3} - \frac{2}{2n-1} \right)$

Telescoping? check to see:

$$\frac{2}{2(n+1)-3} = \frac{2}{2n+2-3} = \frac{2}{2n-1} \therefore \text{ it } \underline{\text{is}} \text{ telescoping.}$$

Converges? check to see:

$$\lim_{n \to \infty} \frac{2}{2n-1} \Rightarrow \frac{2}{\infty} \to 0 \quad \text{This is finite.}$$

$\therefore \boxed{\text{converges}}$

The sum is:

$$\left. \frac{2}{2n-3} \right|_{n=1} = \frac{2}{2(1)-3} = \frac{2}{-1} = -2 \qquad \lim_{n \to \infty} \frac{2}{2n-1} = 0$$

$$\text{so } S = -2 + 0 = \boxed{-2}$$

④. $\sum_{n=1}^{\infty} \frac{(-1)^n (n+3)!}{(2n+1)!}$ Has factorials, so use Ratio Test

Ratio test:

$$\lim_{n = \infty} \left| \frac{\frac{(-1)^{n+1}((n+1)+3)!}{(2(n+1)+1)!}}{\frac{(-1)^n (n+3)!}{(2n+1)!}} \right| = \lim_{n \to \infty} \left| \frac{(-1)^{n+1}(n+4)!}{(2n+3)!} \cdot \frac{(2n+1)!}{(-1)^n (n+3)!} \right|$$

$$= \lim_{n \to \infty} \frac{(n+4)!(2n+1)!}{(2n+3)!(n+3)!}$$

$$= \lim_{n \to \infty} \frac{(n+4)\cancel{(n+3)!}\,\cancel{(2n+1)!}}{(2n+3)(2n+2)\cancel{(2n+1)!}\,\cancel{(n+3)!}}$$

$$= \lim_{n \to \infty} \frac{(n+4)}{(4n^2+10n+6)}$$

$$\stackrel{LH}{=} \lim_{n \to \infty} \frac{1}{8n+10} \to 0$$

$$= 0 < 1$$

$\therefore$ $\boxed{\text{The series converges}}$

~~8~~ 7 $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n(n+1)}{3^n}$ has exponentials so Ratio Test

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Ratio Test

oops I'M out of order

$$\lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+1}(n+2)}{3^{n+1}}}{\frac{(-1)^n(n+1)}{3^n}}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}(n+2)}{3^{n+1}}\cdot\frac{3^n}{(-1)^n(n+1)}\right|$$

$$= \lim_{n\to\infty}\frac{n+2}{3(n+1)}$$

$$= \lim_{n\to\infty}\frac{n+2}{3n+1}$$

$$\overset{L'H}{=} \lim_{n\to\infty}\frac{1}{3} = \frac{1}{3} < 1 \quad \therefore \boxed{\text{The series converges}}$$

6 $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\, n!}{(2n-1)}$

Ratio Test (it has factorial)

$$\lim_{n\to\infty}\left|\frac{\frac{(-1)^n\,(n+1)!}{(2(n+1)-1)}}{\frac{(-1)^{n-1}\,n!}{(2n-1)}}\right| = \lim_{n\to\infty}\left|\frac{(-1)^n(n+1)!}{(2n+1)}\cdot\frac{(2n-1)}{(-1)^{n-1}n!}\right|$$

$$= \lim_{n\to\infty}\frac{(n+1)(2n-1)}{(2n+1)}$$

$$= \lim_{n\to\infty}\frac{2n^2+n-1}{2n+1}$$

$$\overset{L'H}{=} \lim_{n\to\infty}\frac{4n+1}{2}$$

$$\Rightarrow \infty > 1 \longrightarrow \therefore \boxed{\text{The series diverges}}$$

(5) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n (n+5)!}{[2\cdot 7\cdot 12\cdot \ldots \cdot (5n-3)]}$

You're out of order!

Ratio Test:

$$\lim_{n\to\infty}\left|\frac{\dfrac{(-1)^{n+1}(n+6)!}{[2\cdot7\cdot12\cdot\ldots\cdot(5n-3)(5n+2)]}}{\dfrac{(-1)^n(n+5)!}{[2\cdot7\cdot12\cdot\ldots\cdot(5n-3)]}}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}(n+6)!}{[2\cdot7\cdot12\cdot\ldots\cdot(5n-3)(5n+2)]}\cdot\frac{[2\cdot7\cdot12\cdot\ldots\cdot(5n-3)]}{(-1)^n(n+5)!}\right|$$

$$= \lim_{n\to\infty}\frac{(n+6)}{(5n+2)}$$

$$\overset{L'H}{=} \lim_{n\to\infty}\frac{1}{5} = \frac{1}{5} < 1 \quad \therefore \boxed{\text{The series converges}}$$

(8) $\displaystyle\sum_{n=1}^{\infty}\frac{n+5}{n^2\sqrt{n+3}}$ $\Rightarrow$ Ratio of Algebraic functions
$\Rightarrow$ use Limit Comp. Test

Test vs the "fraction of dominant terms"

$$\Rightarrow \frac{n+5}{n^2\sqrt{n}+3} \Rightarrow \frac{n}{n^2\cdot n^{1/2}} \Rightarrow \frac{1}{n\cdot n^{1/2}} \Rightarrow \frac{1}{n^{3/2}} \Rightarrow \text{Use } \sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$$

$\Rightarrow$ p-series $p = \frac{3}{2}$

$p > 1 \Rightarrow$ converges

so we have:

$$\lim_{n\to\infty}\frac{\dfrac{n+5}{n^2\sqrt{n+3}}}{\dfrac{1}{n^{3/2}}} = \lim_{n\to\infty}\frac{(n+5)}{n^2(n+3)^{1/2}}\cdot\frac{n^{3/2}}{1}$$

$$= \lim_{n\to\infty}\frac{n^{5/2}+5n^{3/2}}{(n^4)^{1/2}(n+3)^{1/2}}$$

$$= \lim_{n\to\infty}\frac{n^{5/2}+5n^{3/2}}{(n^5+3n^4)^{1/2}}$$

use ratio of dominant terms

$$= \lim_{n\to\infty}\frac{n^{5/2}}{n^{5/2}}$$

$$= \lim_{n\to\infty} 1 = 1$$ finite & positive

$\therefore$ The series agree.

$\therefore \boxed{\text{The given series converges}}$

(9) $\sum_{n=0}^{\infty}\left(\frac{1}{\sum_{n=0}^{\infty}\left(\frac{2}{5}\right)^n}\right)^n$

$\sum_{n=0}^{\infty}\left(\frac{2}{5}\right)^n \Rightarrow$ geometric with $a=1$ $r=2/5$

$|r|=|2/5|=2/5<1$

$\therefore$ it converges

the sum is

$\sum_{n=0}^{\infty}\left(\frac{2}{5}\right)^n = \frac{1}{1-2/5} = \frac{1}{3/5} = 5/3$

so we have

$\sum_{n=0}^{\infty}\left(\frac{1}{\frac{5}{3}}\right)^n = \sum_{n=0}^{\infty}\left(\frac{3}{5}\right)^n \Rightarrow$ geometric with $a=1$ $r=3/5$

$|r|=|3/5|=3/5<1$

$\therefore$ it converges

the sum is

$\sum_{n=0}^{\infty}\left(\frac{3}{5}\right)^n = \frac{1}{1-3/5} = \frac{1}{2/5} = \boxed{5/2}$

(10) $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\, n}{n^2+1}$

Alternating series test.

Alternating portion is

The "Non-alternating portion" $\frac{n}{n^2+1}$

(i) $\lim_{n\to\infty}\frac{n}{n^2+1} \overset{L'H}{=} \lim_{n\to\infty}\frac{1}{2n} = 0$

(ii) $f(x)=\frac{x}{x^2+1} \Rightarrow f'(x)=\frac{(x^2+1)(1)-(x)(2x)}{(x^2+1)^2}$

$= \frac{x^2+1-2x^2}{(x^2+1)^2}$

$= \frac{-x^2+1}{(x^2+1)^2}$

as $x\to\infty$ the numerator is Negative and the denominator is positive

$\therefore f'(x)<0 \Rightarrow f \searrow$

Since the "Non-alternating portion" has limit $=0$ and is decreasing

$\Rightarrow$ **The given series Converges**

Extra problems from earlier on this sheet

(1)(b) $\sum_{n=3}^{\infty} 12\left(-\frac{2}{3}\right)^n \Rightarrow$ geometric with $a = 12;\ r = -2/3$

$|r| = |-2/3| = 2/3 < 1 \leftarrow$ converges $\leftarrow$

the sum:

$$\sum_{n=0}^{\infty} 12(-2/3)^n = \sum_{n=0}^{2} 12(-2/3)^n + \sum_{n=3}^{\infty} 12(-2/3)^n$$

$$\Rightarrow \frac{a}{1-r} = 12(-2/3)^0 + 12(-2/3)^1 + 12(-2/3)^2 + X$$

$$\Rightarrow \frac{12}{1-(-2/3)} = 12 - 8 + \frac{16}{3} + X$$

$$\frac{12}{1+2/3} = \frac{12}{5/3} = \frac{36}{5}$$

$$\Rightarrow \frac{36}{5} = 12 - 8 + \frac{16}{3} + X$$

$$\Rightarrow \frac{36}{5} = \frac{36-24+16}{3} + X$$

$$\Rightarrow \frac{36}{5} = \frac{28}{3} + X$$

$$\Rightarrow \frac{36}{5} - \frac{28}{3} = X$$

$$\Rightarrow \frac{108-140}{15} = X$$

$$\Rightarrow \boxed{\frac{-32}{15} = X}$$

(1)(c) $\sum_{n=0}^{\infty} 6\left(\frac{7}{3}\right)^n$

$\Rightarrow$ geometric with $a = 6 \quad r = 7/3$

$|r| = |7/3| = 7/3 > 1 \leftarrow$

$\boxed{\therefore \text{DIVERGES}}$

EXTRA TRIG EQUAITONS TO SOLVE – MULTIPLE ANGLE ARGUMENTS

SOLVE EQCH EQUATION IN THE INTERVAL $[0, 2\pi)$

1. $10\cos(x) + 2 = -3$

2. $2\sin(x) = 1$

3. $\tan(x) + 1 = 0$

4. $4\cos^2(x) - 10 = -7$

5. $3\tan^2(x) + 1 = 4$

6. $4\cos^2(x) - 2 = 0$ Hint: Both of $\sqrt{\frac{2}{4}}$ and $\sqrt{\frac{1}{2}}$ are equal to $\frac{\sqrt{2}}{2}$

7. $3\tan(x) - 1 = 0$

8. $2\cos(x) = 1$

9. $2\sin(x) + 5\sqrt{3} = 4\sqrt{3}$

10. $\tan^2(x) - 3 = 0$

11. $2\sin(x) + \sqrt{2} = 0$