Solutions

(I)

1. C
2. E There are 2.
3. D
4. B
5. C
6. D
7. C
8. E
9. C (odd multiples of $\pi/2$)
10. A
11. B
12. E The limit is $\pi+1$
13. E
14. C substitute & simplify

(II) 20. $f(x) = \begin{cases} \dfrac{-3}{x^2-4} & \text{if } x \le 1 \\ 3-x^2 & \text{if } x > 1 \end{cases}$

* $\dfrac{-3}{x^2-4}$ has discontinuity at $x=-2$ and $x=2$
However, $x=2$ is NOT in the region $x \le 1$
so the discontinuity is at $x=-2$

* we must investigate $x=1$ the change of behavior value independently

consider at $x=1$

(i) $f(1) \overset{TOP}{=} \dfrac{-3}{(1)^2-4} = \dfrac{-3}{1-4} = \dfrac{-3}{-3} = 1$

(ii) $\lim_{x\to 1} f(x)$ 2 → (a) $\lim_{x\to 1^-} f(x) \overset{TOP}{=} \lim_{x\to 1^-} \left(\dfrac{-3}{(1)^2-4}\right) = \dfrac{-3}{-3} = 1$

(b) $\lim_{x\to 1^+} f(x) \overset{BOT}{=} \lim_{x\to 1^+} (3-(1)^2) = 3-1 = 2$

since these do not equal $\lim_{x\to 1} f(x)$ D.N.E.

Thus Discontinuities at $X=-2$ and $X=1$.

(21). $f(x) = \begin{cases} x^2 - kx & \text{if } x < -2 \\ 4x+2 & \text{if } x \geq -2 \end{cases}$

f will be continuous if $\lim_{x\to(-2)^-} f(x) = \lim_{x\to(-2)^+} f(x)$

TOP

so we require $\Rightarrow \lim_{x\to(-2)^-} (x^2-kx) = \lim_{x\to(-2)^+} (4x+2)$

$\Rightarrow (-2)^2 - k(-2) = 4(-2)+2$

$\Rightarrow 4 + 2k = -8+2$

$\Rightarrow 4+2k = -6$

$\Rightarrow 2k = -10$

$\Rightarrow \boxed{k = -5}$

(22) $\lim_{x\to 3} \frac{(\sqrt{x+1}-2)}{(x-3)} = \lim_{x\to3} \frac{(\sqrt{x+1}-2)}{(x-3)} \cdot \frac{(\sqrt{x+1}+2)}{(\sqrt{x+1}+2)}$

$= \lim_{x\to3} \frac{(x+1-4)}{(x-3)(\sqrt{x+1}+2)}$

$= \lim_{x\to3} \frac{\cancel{(x-3)}}{\cancel{(x-3)}(\sqrt{x+1}+2)}$

$= \lim_{x\to3} \frac{1}{(\sqrt{x+1}+2)}$

$= \frac{1}{\sqrt{(3)+1}+2}$

$= \boxed{\frac{1}{4}}$

(23) $f(x) = \begin{cases} -\cos(x) & \text{if } x<0 \\ \tan(x) & \text{if } x \geq 0 \end{cases}$ on $(-2\pi, 2\pi)$

(*) Note that $-\cos(x)$ has no discontinuities on $(-2\pi, 0)$, and that $\tan(x)$ has discontinuities at $x = \pi/2$ & $x = 3\pi/2$ (vertical asymptotes).

(*) ⇓ NOTE!! YOU NEED TO KNOW THE DOMAINS OF ALL THE TRIG FUNCTIONS!!

At the "change of behavior value" $x=0$ we must check the 3-point definition of continuity.

(i) $f(0) \overset{\text{Bot}}{=} \tan(0) = 0$

(ii) $\lim\limits_{x\to 0} f(x)$

(a) $\lim\limits_{x\to 0^-} f(x) \overset{\text{Top}}{=} \lim\limits_{x\to 0^-} (-\cos(x)) = -\cos(0) = -1$

(b) $\lim\limits_{x\to 0^+} f(x) \overset{\text{Bot}}{=} \lim\limits_{x\to 0^+} (\tan(x)) = \tan(0) = 0$

since these do not equal, $\lim\limits_{x\to 0} f(x)$ D.N.E.

∴ There is a discontinuity at $x=0$

∴ The discontinuities are at

$\boxed{x=0,\ x=\pi/2,\ x=3\pi/2}$

(24). $\lim\limits_{x\to 4} \dfrac{x^2-6x+8}{x^2-x-12} = \lim\limits_{x\to 4} \dfrac{\cancel{(x-4)}(x-2)}{(x+3)\cancel{(x-4)}}$

$= \lim\limits_{x\to 4} \dfrac{(x-2)}{(x+3)}$

$= \dfrac{(4)-2}{(4)+3} = \boxed{\dfrac{2}{7}}$

(25) omit

(26) $f(x) = \begin{cases} 4x - x^2 & \text{if } x \le -1 \\ \dfrac{15}{x+4} & \text{if } x > -1 \end{cases}$

(i) $f(-1) \overset{TOP}{=} 4(-1) - (-1)^2$
$= -4 - 1$
$= -5$

(ii) $\lim_{x \to (-1)} f(x)$ (a) $\lim_{x \to (-1)^-} f(x) \overset{TOP}{=} \lim_{x \to (-1)^-} (4x - x^2) = 4(-1) - (-1)^2 = -4 - 1 = -5$

(b) $\lim_{x \to (-1)^+} f(x) \overset{BOT}{=} \lim_{x \to (-1)^+} \left(\frac{15}{x+4}\right) = \frac{15}{(-1)+4} = \frac{15}{3} = 5$

since these do NOT equal $\lim_{x \to (-1)} f(x)$ D.N.E.

$\therefore f(x)$ is NOT continuous at $x = -1$

(27) Prove $\lim_{x \to (-3)} (-5x - 6) = 9$

(i) Let $\epsilon > 0$

(ii) Search for δ:

$|f(x) - L| < \epsilon \Rightarrow |(-5x - 6) - 9| < \epsilon$

$\Rightarrow |-5x - 15| < \epsilon$

$\Rightarrow |-5(x+3)| < \epsilon$

$\Rightarrow |-5||x - (-3)| < \epsilon$

$\Rightarrow |x - (-3)| < \frac{\epsilon}{|-5|}$

choose $\delta = \frac{\epsilon}{|-5|}$

(iii) Verify δ:

$0 < |x - a| < \delta \Rightarrow$

$\Rightarrow |x - (-3)| < \frac{\epsilon}{|-5|}$

$\Rightarrow |-5||x + 3| < \epsilon$

$\Rightarrow |(-5)(x+3)| < \epsilon$

$\Rightarrow |-5x - 15| < \epsilon$

$\Rightarrow |-5x - 6 + 6 - 15| < \epsilon$

$\Rightarrow |(-5x - 6) - 9| < \epsilon$

$\Rightarrow |f(x) - L| < \epsilon$