Solutions

① $\int \cos^2(x)\,dx = \frac{1}{2}\int [1+\cos(2x)]\,dx$

$= \boxed{\frac{1}{2}\left[x + \frac{1}{2}\cos(2x)\right] + C}$

② $\int 8x^2 \sin(2x)\,dx$

parts: $u = 8x^2$ $\quad v = -\frac{1}{2}\cos(2x)$

$du = 16x\,dx \quad dv = \sin(2x)\,dx$

$= -\left(\frac{1}{2}\right)(8x^2)\cos(2x) - \int 16x\left(-\frac{1}{2}\cos(2x)\right)dx$

$= -4x^2\cos(2x) + \int 8x\cos(2x)\,dx$

↳ work this separately

parts: $u = 8x \quad v = -\frac{1}{2}\sin(2x)$

$du = 8\,dx \quad dv = \cos(2x)\,dx$

$= -\frac{1}{2}(8x)\sin(2x) - \int \left(-\frac{1}{2}\right)(8)\sin(2x)\,dx$

$= -4\sin(2x) + 4\int \sin(2x)\,dx$

$= -4\sin(2x) + 4\left(-\frac{1}{2}\right)\cos(2x) + C$

so we have:

$\boxed{-4x^2\cos(2x) - 4\sin(2x) - 2\cos(2x) + C}$

③ $\int \sin^5(x)\cos^2(x)\,dx = \int \sin^4(x)\cdot\cos^2(x)\cdot \underline{\sin(x)\,dx}$

$= \int [1-\cos^2(x)]^2 \cdot \cos^2(x)\,\underline{\sin(x)\,dx}$

$= \int [1-2\cos^2(x)+\cos^4(x)] \cdot \cos^2(x) \cdot \underline{\sin(x)\,dx}$

$= \int [\cos^2(x) - 2\cos^4(x) + \cos^6(x)]\,\underline{\sin(x)\,dx}$

Let $u = \cos(x)$

$\Rightarrow du = -\sin(x)\,dx$

$\Rightarrow -du = \underline{\sin(x)\,dx}$

$= -\int [u^2 - 2u^4 + u^6]\,du$

$= -\left[\frac{u^3}{3} - \frac{2u^5}{5} + \frac{u^7}{7}\right] + C$

$= \boxed{-\left[\frac{\cos^3(x)}{3} - \frac{2\cos^5(x)}{5} + \frac{\cos^7(x)}{7}\right] + C}$

④ $\int \tan^3(x)\sec^4(x)\,dx = \int \tan^3(x)\cdot\sec^2(x)\,\underline{\sec^2(x)\,dx}$

$= \int \tan^3(x)\left[\tan^2(x)+1\right]\cdot\underline{\sec^2(x)\,dx}$

$= \int [\tan^5(x) + \tan^3(x)]\,\underline{\sec^2(x)\,dx}$

Let $u = \tan(x)$

$du = \underline{\sec^2(x)\,dx}$

$= \int [u^5 + u^3]\,du$

$= \frac{u^6}{6} + \frac{u^4}{4} + C$

$= \boxed{\frac{\tan^6(x)}{6} + \frac{\tan^4(x)}{4} + C}$

#4 continued

#4 continued

Note: #4 can be done two ways. Here is the other

$$\int \tan^3(x)\sec^4(x)\,dx = \int \tan^2(x)\cdot\sec^3(x)\cdot \sec(x)\tan(x)\,dx$$

$$= \int \left[\sec^2(x)-1\right]\cdot\sec^3(x)\cdot \sec(x)\tan(x)\,dx$$

$$= \int \left[\sec^5(x)-\sec^3(x)\right]\cdot \sec(x)\tan(x)\,dx$$

Let $u = \sec(x)$

$\Rightarrow du = \sec(x)\tan(x)\,dx$

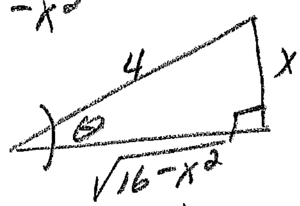
$$= \int \left[u^5-u^3\right]du$$

$$= \frac{u^6}{6} - \frac{u^4}{4} + C$$

$$= \boxed{\frac{\sec^6(x)}{6} - \frac{\sec^4(x)}{4} + C}$$

⑤ $\displaystyle\int \frac{x^3}{16\sqrt{16-x^2}}\,dx$

Trig sub:



choose $\sin(\theta) = \frac{x}{4}$

$\Rightarrow 4\sin(\theta) = x$

$\Rightarrow 4\cos(\theta)\,d\theta = dx$

choose $\cos(\theta) = \frac{\sqrt{16-x^2}}{4}$

$\Rightarrow 4\cos(\theta) = \sqrt{16-x^2}$

$$= \int \frac{(4\sin(\theta))^3\cdot 4\cos(\theta)}{16\cdot 4\cos(\theta)}\,d\theta$$

$$= \frac{64}{16}\int \sin^3(\theta)\,d\theta$$

$$= 4\int \sin^2(\theta)\cdot\sin(\theta)\cdot d\theta$$

$$= 4\int \left[1-\cos^2(\theta)\right]\cdot\sin(\theta)\,d\theta$$

Let $u = \cos(\theta)\,d\theta$

$\Rightarrow du = -\sin(\theta)\,d\theta$

$\Rightarrow -du = \sin(\theta)\,d\theta$

$$= -4\int\left[1-u^2\right]du$$

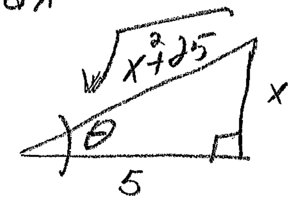
$$= -4\left[u - \frac{u^3}{3}\right] + C$$

$$= -4\left[\cos(\theta) - \frac{\cos^3(\theta)}{3}\right] + C$$

$$= \boxed{-4\left[\frac{\sqrt{16-x^2}}{4} - \frac{1}{3}\left(\frac{\sqrt{16-x^2}}{4}\right)^3\right] + C}$$

6. $\int \frac{x^2}{\sqrt{x^2+25}}\,dx$

trig sub



choose

$\tan(\theta) = \frac{x}{5}$

$\Rightarrow 5\tan(\theta) = x$

$\Rightarrow 5\sec^2(\theta)\,d\theta = dx$

choose

$\sec(\theta) = \frac{\sqrt{x^2+25}}{5}$

$\Rightarrow 5\sec(\theta) = \sqrt{x^2+25}$

$$= \int \frac{(5\tan(\theta))^2 \cdot \cancel{5}\sec^{\cancel{2}}(\theta)}{\cancel{5}\sec(\theta)}\,d\theta$$

$$= \int 25\tan^2(\theta)\cdot\sec(\theta)\,d\theta$$

$$= 25\int\left[\sec^2(\theta) - 1\right]\cdot\sec(\theta)\,d\theta$$

$$= 25\int\left[\sec^3(\theta) - \sec(\theta)\right]d\theta$$

$$= 25\left[\frac{1}{2}\left[\ln\left|\sec(\theta)+\tan(\theta)\right| + \sec(\theta)\tan(\theta)\right]\right] + C$$

$$= \boxed{\frac{25}{2}\left[\ln\left|\frac{\sqrt{x^2+25}}{5} + \frac{x}{5}\right| + \left(\frac{\sqrt{x^2+25}}{5}\right)\left(\frac{x}{5}\right)\right] + C}$$

7. $\int \frac{6x^2+4}{x^3+3x^2-4x}\,dx = \int \frac{6x^2+4}{x(x^2+3x-4)}\,dx$

$$= \int \frac{6x^2+4}{x(x+4)(x-1)}\,dx$$

Par Frac

$$x(x+4)(x-1)\left[\frac{6x^2+4}{x(x+4)(x-1)} = \frac{A}{x} + \frac{B}{(x+4)} + \frac{C}{(x-1)}\right]$$

$$\Rightarrow 6x^2+4 = A(x+4)(x-1) + B(x)(x-1) + C(x)(x+4)$$

Let $x=0$

$\Rightarrow 0+4 = A(0+4)(0-1) + 0 + 0$

$\Rightarrow 4 = A(4)(-1)$

$\Rightarrow 4 = A(-4)$

$\Rightarrow A = -1$

Let $x=-4$

$\Rightarrow 6(-4)^2+4 = 0 + B(-4)(-4-1) + 0$

$\Rightarrow 100 = B(20)$

$\Rightarrow B = 10$

cont

$\Downarrow$

Let $x=1$

$\Rightarrow 6(1)^2+4 = 0+0+C(1)(1+4)$

$\Rightarrow 10 = C(5)$

$\Rightarrow C=2$

so we have:

$$\int \frac{-1}{x}\,dx + \int \frac{10}{x+4}\,dx + \int \frac{2}{x-1}\,dx = \boxed{-\ln|x| + 10\ln|x+4| + 2\ln|x-1| + C}$$

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(8) $\int \frac{x^2-x+2}{(x+1)(x-1)^2}\,dx$

Par Frac

$$\frac{(x+1)(x-1)^2}{1}\left[\frac{x^2-x+2}{(x+1)(x-1)^2} = \frac{A}{(x+1)} + \frac{B}{(x-1)} + \frac{C}{(x-1)^2}\right]$$

$\Rightarrow x^2-x+2 = A(x-1)^2 + B(x+1)(x-1) + C(x+1)$

Let $x=1$

$\Rightarrow (1)^2-1+2 = A(0)+B(0)+C(0+1)$

$\Rightarrow 2 = C$

Let $x=-1$

$\Rightarrow (-1)^2-(-1)+2 = A((-1)-1)^2+0+0$

$\Rightarrow 1+1+2 = A(4)$

$\Rightarrow A=1$

Let $x=0$

$\Rightarrow 2 = (1)(0-1)^2 + B(0+1)(0-1) + (2)(0+1)$

$\Rightarrow 2 = 1 + B(-1) + 2$

$\Rightarrow B=1$

$\Downarrow$ cont. $\Rightarrow$

cont.

so we have

$$\int \frac{1}{x+1}dx + \int \frac{1}{x-1}dx + \int \frac{2}{(x-1)^2}dx$$

work this separately

$$= \int 2(x-1)^{-2}dx = \frac{2(x-1)^{-1}}{-1} + C$$

$$= -2(x-1)^{-1} + C$$

so we obtain:

$$\boxed{= \ln|x+1| + \ln|x-1| - 2(x-1)^{-1} + C}$$

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2. $\displaystyle\int \frac{7x^2+12}{x^3+4x}dx = \int \frac{7x^2+12}{x(x^2+4)}dx$ TYPE III

Par Frac:

$$\frac{x(x^2+4)}{1}\left[\frac{7x^2+12}{x(x^2+4)} = \frac{A}{x} + \frac{(Bx+C)}{(x^2+4)}\right]$$

$$\Rightarrow 7x^2+12 = A(x^2+4) + (Bx+C)(x)$$

$$= Ax^2 + 4A + Bx^2 + Cx$$

$$= (A+B)x^2 + Cx + 4A$$

EQUATE COEFFICIENTS

$\Rightarrow A+B = 7 \Rightarrow B=3$

$C = 0 \Rightarrow C=0$

$4A = 12 \Rightarrow A=3$

so we have

$$\int \frac{3}{x}dx + \int \frac{3x}{x^2+4}dx$$

separately

Let $u = x^2+4$

$\Rightarrow du = 2x\,dx$

$\Rightarrow \frac{1}{2}u = x\,dx$

$\Rightarrow \frac{3}{2}du = 3x\,dx$

$= \frac{3}{2}\int \frac{1}{u}du$

$= \ln|u| + c = \ln|x^2+4| + c$

So:

$$\boxed{= 3\ln|x| + \frac{3}{2}\ln|x^2+4| + C}$$

pg Zeven

(10) $\lim_{x\to 0} \frac{\cos(x)-1}{e^x-1}$ ⇒ direct subs. gives "$\frac{0}{0}$"

$\overset{L'H}{=} \lim_{x\to 0} \frac{-\sin(x)}{e^x}$

$= \frac{0}{1}$

$= \boxed{0}$.

(11) $\lim_{x\to 0} \frac{(\sin(x)-x)}{(e^x-x-1)}$ ⇒ direct subs gives "$\frac{0}{0}$"

$\overset{L'H}{=} \lim_{x\to 0} \frac{\cos(x)-1}{e^x-1}$ ⇒ again "$\frac{0}{0}$"

$\overset{L'H}{=} \lim_{x\to 0} \frac{-\sin(x)}{e^x}$

$= \frac{0}{1}$

$= \boxed{0}$.

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(12) $\int_2^\infty \frac{1}{(x-1)^4}\,dx = \lim_{b\to\infty}\left[\int_2^b (x-1)^{-4}\,dx\right]$

$= \lim_{b\to\infty}\left[\frac{(x-1)^{-3}}{-3}\right]_2^b$

$= \lim_{b\to\infty}\left[\left(\frac{-1}{3(x-1)^3}\right)\right]_2^b$

$= \lim_{b\to\infty}\left[\left(\cancel{\frac{-1}{3(b-1)^3}}^{\,0}\right)-\left(\frac{-1}{3(2-1)^3}\right)\right]$

$= 0 + \frac{1}{3}$

$= \boxed{\frac{1}{3}}$.