11.7: Power Series

A power series is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

where x is a variable and the c_n 's are constants called the coefficients of the series. For each fixed value of x, the series is then a series of constants that can be tested for convergence or divergence. The sum of the series is a function

$$f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots.$$

Its domain is the set of all x for which the series converges. Notice that this function resembles a polynomial function, except that it has an infinite number of terms.

A more general form is the following:

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + c_3 (x-a)^3 + \cdots$$

This is called a **power series in** $(x-a)$, or a **power series centered at *a***, or a **power series about *a***. Notice that for $x = a$, each term is 0, and so such a series will always converge for at least $x = a$.

Example 1: For what values of x is the series $\sum_{n=0}^{\infty} \frac{n!(x-2)^n}{5^n}$ convergent?

Example 2: For what values of *x* is the series $\sum_{n=0}^{\infty}(-1)^n 2^{n+1}(x+3)^n$ convergent?

Theorem For a given power series $\sum_{n=0}^{\infty} c_n (x-a)^n$, there are only three possibilities: 1. The series converges only for $x=a$. 2. The series converges for all *x*. 3. There is a positive number *R* such that the series converges for $\lvert x-a\rvert < R$ and diverges for $\lvert x-a\rvert > R$.

The number *R* is called the **radius of convergence** of the power series. By convention, $R=0$ in case 1, and $R=\infty$ for case 2.

The **interval of convergence** of a power series consists of all values of x for which the series converges. For case 1, the interval is a single point *a*. For case 2, the interval is $(-\infty,\infty)$. For case 3, the interval is $a-R<x<a+R$. For the endpoints $x=a\pm R$, anything could happen. That is, the series may or may not converge at either endpoint.

Example 3: Find the radius and interval of convergence for the geometric power series $\sum_{n=0}^{\infty} x^n$. Find its sum within the interval of convergence.

Example 4: Find the radius and interval of convergence for the series $\sum_{n=0}^{\infty}\left(\frac{x^3-1}{2}\right)^n$. Find its sum within the interval of convergence.

<u>Example 5:</u> Find the radius and interval of convergence for the series $\sum_{n=1}^{\infty}\frac{(x-3)^n}{n\cdot 3^n}$.

Representations of functions as power series:

The geometric series

$$\sum_{n=0}^{\infty}x^n = 1+x+x^2+x^3+\cdots$$

is convergent when $|x|<1$. In fact, we have

$$\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n = 1+x+x^2+x^3+\cdots$$

for $|x|<1$.

This form allows us to write certain rational functions as sums of geometric series.

<u>Example 1:</u> Express $\frac{2}{1+x^3}$ as the sum of a power series and find the interval of convergence.

Example 2: Find a power series representation for $\frac{1}{x+4}$ and find the interval of convergence.

Example 3: Find a power series representation for $\frac{x^2}{x+4}$ and find the interval of convergence.

Differentiation and Integration of Power Series:

The sum of a power series is a function $f(x)=\sum_{n=0}^{\infty} c_n (x-a)^n$ whose domain is the interval of convergence of the series. The following theorem allows us to differentiate and integrate such functions by what is called **term-by-term differentiation and integration**.

Theorem

If the power series $\sum_{n=0}^{\infty} c_n (x-a)^n$ has a radius of convergence $R > 0$, then the function defined by

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots = \sum_{n=0}^{\infty} c_n (x-a)^n$$

is differentiable and continuous on the interval $(a-R, a+R)$ and:

- $f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \cdots = \sum_{n=1}^{\infty} nc_n (x-a)^{n-1}$
- $\int f(x)\,dx = C + c_0(x-a) + c_1\dfrac{(x-a)^2}{2} + c_2\dfrac{(x-a)^3}{3} + c_3\dfrac{(x-a)^4}{4} + \cdots = C + \sum_{n=0}^{\infty} c_n \dfrac{(x-a)^{n+1}}{n+1}$

The radii of convergence of the two power series above are both *R*.

NOTE: Although the radius of convergence remains the same when the series is differentiated or integrated, this does NOT mean the interval of convergence remains the same. It is possible that the original series converges at an endpoint while the differentiated series does not. It is possible that the integrated series converges at an endpoint while the original series does not.

Example 4: Express $\dfrac{1}{(x+4)^2}$ as a power series by differentiating $\dfrac{1}{x+4}$ from a previous example. What is the radius of convergence?

Example 5: Find a power series for $\ln(x+4)$ and its radius of convergence.