11.8, 11.9: Taylor and Maclaurin Series

Although we were able to find power series representations for a limited group of functions in the previous section, it is not immediately obvious whether any given function has a power series representation. Which functions have a power series representation, and how can we find such a representation?

Suppose that *f* is any function that can be represented by a power series.

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + c_4(x-a)^4 + \cdots \qquad |x-a| < R$$

First notice that if $x = a$, all terms after the first term are 0 and we have

$$f(a) = c_0$$

Differentiating *f*, we have

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + 5c_5(x-a)^4 + \cdots \qquad |x-a| < R$$

Again, letting $x = a$, all terms after the first term are 0 and we have

$$f'(a) = c_1$$

Differentiating *f* again, we have

$$f''(x) = 2c_2 + (2\cdot 3)c_3(x-a) + (3\cdot 4)c_4(x-a)^2 + (4\cdot 5)c_5(x-a)^3 + (5\cdot 6)c_6(x-a)^4 + \cdots \quad |x-a| < R$$

Again, letting $x = a$, all terms after the first term are 0 and we have

$$f''(a) = 2c_2$$

Differentiating *f* once more, we have

$$f'''(x) = (2\cdot 3)c_3 + (2\cdot 3\cdot 4)c_4(x-a) + (3\cdot 4\cdot 5)c_5(x-a)^2 + (4\cdot 5\cdot 6)c_6(x-a)^3 + (5\cdot 6\cdot 7)c_7(x-a)^4 + \cdots$$

$$|x-a| < R$$

Again, letting $x = a$, all terms after the first term are 0 and we have

$$f'''(a) = (2\cdot 3)c_3 = 3!c_3$$

So, here is the pattern: substituting $x = a$ in the *n*th derivative $f^{(n)}(x)$ to obtain

$$f^{(n)}(a) = n!c_n$$

Solving this equation for the coefficient we have

$$c_n = \frac{f^{(n)}(a)}{n!}$$

This formula is valid even for $n = 0$ because $0! = 1$ and if we adopt the notation $f^{(0)} = f$.

Theorem

If *f* has a power series representation (expansion) at *a*, that is, if

$$f(x)=\sum_{n=0}^{\infty} c_n (x-a)^n, \quad |x-a|<R$$

then its coefficients are determined by the formula

$$c_n = \frac{f^{(n)}(a)}{n!}$$

Putting these coefficient values back into the series we see that if *f* has a power series expansion at *a*, then it must have the form

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \\ &= f(a)+\frac{f'(a)}{1!}(x-a)+\frac{f''(a)}{2!}(x-a)^2+\frac{f'''(a)}{3!}(x-a)^3+\cdots \end{aligned}$$

The series in this theorem is called the **Taylor series of the function *f* at *a*** (or **about *a***, or **centered at *a***). When $a=0$, the series becomes

$$f(x)=\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0)+\frac{f'(0)}{1!}x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\cdots$$

which is called the **Maclaurin series of the function *f***.

<u>Example 1:</u> Find the Maclaurin series of the function $f(x)=e^x$ and its radius of convergence.

WARNING!!

We must be careful at this point, however. According to the theorem earlier, **if** e^x has a power series expansion at 0, then it must be $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. However, we have not demonstrated that e^x really does have a power series representation.

With what conditions is a function equal to the sum of its Taylor series? As with any convergent series, this means that $f(x)$ is the limit of the sequence of partial sums. For a Taylor series, these partial sums are

$$\begin{aligned} T_n(x) &= \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i \\ &= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n \end{aligned}$$

T_n is called the ***n*th-order (or *n*th degree) Taylor polynomial of *f* at *a***. For the previous example $f(x) = e^x$, we have the following Taylor polynomials at 0 (also called Maclaurin polynomials) for $n = 1, 2, 3$:

$$T_1(x) = 1 + x \qquad T_2(x) = 1 + x + \frac{x^2}{2!} \qquad T_3(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$$

So, $f(x)$ is the sum of its Taylor series if $f(x) = \lim_{n\to\infty} T_n(x)$.

<u>Taylor's Theorem:</u>

If f has derivatives of all orders in an open interval I containing a, then for each positive integer n and for each x in I, then

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x),$$

where $R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$ for some c between a and x.

$R_n(x)$ is called the **remainder** of the Taylor series and $R_n(x) = f(x) - T_n(x)$. If $R_n(x) \to 0$ as $x \to \infty$, then $T_n(x) \to f(x)$ and the Taylor series converges to $f(x)$.

To show that $\lim_{n\to\infty} R_n(x) = 0$ for a specific function f, we often use the following.

Remainder Estimation Theorem (Taylor's Inequality):

If $\left|f^{(n+1)}(t)\right| \le M$ for all t between x and a, inclusive, then the remainder $R_n(x)$ of the Taylor series satisfies the inequality

$$\left|R_n(x)\right| \le \frac{M}{(n+1)!}\left|x-a\right|^{n+1}.$$

Example 2: Find the Maclaurin series for $f(x) = \cos x$.

Example 3: Find the Maclaurin series for the function $f(x) = x^2 \sin x$.

Instead of computing derivatives (which would be messy), take the Maclaurin series for $\sin x$ and multiply by x^2.

Example 4: Find the Taylor polynomials of orders 0, 1, 2, 3, and the Taylor series $f(x) = \cos x$ centered at $\pi/3$. (Put another way: generated by $f(x) = \cos x$ at $\pi/3$)

Example 5: Find the Taylor polynomials of orders 0, 1, 2, 3, and the Taylor series $f(x) = 3x^4 - 2x^3 + 5x + 1$ centered at -2. (Put another way: generated by $f(x)$ at -2.)

Important Maclaurin Series

$\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n=1+x+x^2+x^3+\cdots$	$(-1,1)$
$e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}=1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots$	$(-\infty,\infty)$
$\sin x=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots$	$(-\infty,\infty)$
$\cos x=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots$	$(-\infty,\infty)$
$\tan^{-1}x=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\cdots$	$[-1,1]$

Example 6: Evaluate $\int \sin\left(x^2\right)dx$.

Example 7: Evaluate $\lim_{x\to 0}\frac{\sin x-x}{x^3}$ with a series.