Chapter 10: Infinite Sequences and Series

10.1: Sequences

A sequence can be thought of as a list of numbers in a specified order:

$$a_1, a_2, a_3, a_4, \ldots, a_n, \ldots$$

Each number in the sequence is a term. Sequences can be finite (there is a last term) or infinite. We will be studying infinite sequences exclusively.

A sequence can also be thought of as a function whose domain is the natural numbers. However, we rarely use the function notation for sequences. That is, we will write a_n to represent the *n*th term, rather than $f(n)$.

Because a sequence forms a well-defined set, a sequence is often written using the following notations

$$\{a_1, a_2, a_3, a_4, \ldots, a_n, \ldots\} \quad \text{or} \quad \{a_n\} \quad \text{or} \quad \{a_n\}_{n=1}^{\infty}$$

Many sequences can be defined by a formula for the *n*th term. Also, note that *n* does not always begin with 1.

Example 1: $\left\{\dfrac{n+1}{n}\right\}_{n=1}^{\infty}$ or $a_n = \dfrac{n+1}{n}$ or $\left\{\dfrac{2}{1}, \dfrac{3}{2}, \dfrac{4}{3}, \dfrac{5}{4}, \ldots, \dfrac{n+1}{n}, \cdots\right\}$

Example 2: $\left\{(-1)^{n+1}(2n+1)\right\}_{n=0}^{\infty}$ or $a_n = (-1)^{n+1}(2n+1)$

or $\left\{-1, 3, -5, 7, \ldots, (-1)^{n+1}(2n+1), \cdots\right\}$

Example 3: Find a formula for the general term a_n of the sequence $\{-1, 4, -9, 16, -25, \ldots\}$.

Not all sequences can be defined with a simple formula for the *n*th term.

Example 4: The sequence $\{p_n\}$ where p_n represents the *n*th decimal place of the number π.

Graphs:

Sequences can be graphed in two different ways. One method is to plot the numbers on a real line. Another method is to plot the ordered pairs (n, a_n) in the plane.

Example 5: Graph the sequence $a_n = \dfrac{1}{n}$.

From these graphs, it appears that the terms of the sequence approach 0 as *n* becomes large. In other words, we can write

$$\lim_{n\to\infty} \frac{1}{n} = 0$$

<u>Definition of Limit (Informal)</u>

A sequence $\{a_n\}$ has the **limit** *L* and we write

$$\lim_{n\to\infty} a_n = L \qquad \text{or} \qquad a_n \to L \text{ as } n \to \infty$$

if we can make the terms a_n as close to *L* as we like by taking *n* to be sufficiently large.

If the limit exists, we say the sequence **converges**. Otherwise, we say the sequence **diverges**.

<u>Definition of Limit (Formal)</u>

A sequence $\{a_n\}$ has the **limit** *L* and we write

$$\lim_{n\to\infty} a_n = L \qquad \text{or} \qquad a_n \to L \text{ as } n \to \infty$$

if for every $\varepsilon > 0$ there is a corresponding integer *N* such that

$$|a_n - L| < \varepsilon \qquad \text{whenever} \qquad n > N .$$

Notice how similar the limit definition is to the definition of a limit for a function (Section 4.4). Because of this similarity, we have the following theorem.

Theorem:

If $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$ when n is an integer, then $\lim_{n\to\infty} a_n = L$.

If a_n becomes large as n becomes large, then we say $\lim_{n\to\infty} a_n = \infty$ and that $\{a_n\}$ diverges to ∞.

Limit Laws

Let $\{a_n\}$ and $\{b_n\}$ be convergent sequences and let *c* be a constant. Then, the following are true.

- $\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$
- $\lim_{n\to\infty}(a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n$
- $\lim_{n\to\infty} ca_n = c\lim_{n\to\infty} a_n$
- $\lim_{n\to\infty} c = c$
- $\lim_{n\to\infty}(a_n b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$
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- $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$, provided $\lim_{n\to\infty} b_n \neq 0$
- $\lim_{n\to\infty} a_n^p = \left(\lim_{n\to\infty} a_n\right)^p$, provided $p > 0$ and $a_n > 0$

Theorem:

$\lim_{n\to\infty}|a_n| = 0$ if and only if $\lim_{n\to\infty} a_n = 0$.

The Sandwich (or Pinching or Squeeze) Theorem (for sequences):

If $a_n \le b_n \le c_n$ for $n \ge n_0$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.

An Important Limit: $\lim_{n\to\infty} \frac{x^n}{n!} = 0$

The above limit can be proved using the Sandwich Theorem.

Example 6: Evaluate $\lim\limits_{n\to\infty}\left(\dfrac{2n^2-n-6}{n^2}\right)$.

Example 7: Determine whether the sequence $a_n=(-1)^{n+1}$ converges or diverges.

Example 8: Evaluate $\lim\limits_{n\to\infty}\dfrac{(-1)^n n^2}{n^3+4}$.

The sequence $\{r^n\}$ is convergent if $-1<r\le 1$ and divergent for all other values of *r*. In particular,

$$\lim_{n\to\infty} r^n=\begin{cases}0 & \text{if } -1<r<1\\ 1 & \text{if } r=1\end{cases}$$

Monotonic sequences:

A sequence $\{a_n\}$ is called **increasing** if $a_n < a_{n+1}$ for all $n \geq 1$. It is called **decreasing** if $a_n > a_{n+1}$ for all $n \geq 1$. It is called **monotonic** if it is either increasing or decreasing.

Example 9: Determine whether the given sequence is monotonic. If it is, determine whether it is increasing or decreasing.

(a) $a_n = \dfrac{n}{n^2+n}$ (b) $a_n = \tan\left(\dfrac{(2n-1)\pi}{4}\right)$

Bounded sequences:

A sequence $\{a_n\}$ is *bounded above* if there is a number *M* such that $a_n \leq M$ for all $n \geq 1$.
A sequence is *bounded below* if there is a number *m* such that $m \leq a_n$ for all $n \geq 1$.
If a sequence is bounded above and below, then $\{a_n\}$ is a *bounded sequence.*

Example 10: $a_n = -n^2$ is bounded above $a_n \leq 0$, but it is not bounded below.

Example 11: $a_n = \dfrac{n}{n^2+n}$

Monotonic Sequence Theorem

Every bounded, monotonic sequence is convergent.