10.2: Infinite Series

Given a sequence $\{a_n\}_{n=1}^{\infty}$, the sum of the sequence has the form

$$a_1 + a_2 + a_3 + \ldots + a_n + \ldots$$

and is called an **infinite series** (or just simply a **series**). This series is denoted by

$$\sum_{n=1}^{\infty} a_n \quad \text{or} \quad \sum a_n$$

However, does it make sense to have a sum of an *infinite* number of terms?

Example 1: There is no finite sum for the series defined by the sequence $a_n = n^2$

$$1 + 4 + 9 + 16 + 25 + \ldots$$

Example 2: Consider the series below.

$$\frac{2}{3} + \frac{2}{9} + \frac{2}{27} + \frac{2}{81} + \ldots + \frac{2}{3^n} + \ldots$$

If we calculate the sums of the first few terms we obtain

$$\frac{2}{3}, \frac{8}{9}, \frac{26}{27}, \frac{80}{81}, \ldots, 1 - \frac{1}{3^n}, \ldots$$

As we add more terms, the *partial sums* get closer to 1. So, we write

$$\sum_{n=1}^{\infty} \frac{2}{3^n} = 1.$$

Determining whether a series has a finite sum (converges):

We use **partial sums** s_n of the sequence of terms.

$$s_1 = a_1$$
$$s_2 = a_1 + a_2$$
$$s_3 = a_1 + a_2 + a_3$$
$$s_4 = a_1 + a_2 + a_3 + a_4$$
$$\vdots$$
$$s_n = a_1 + a_2 + a_3 + \ldots + a_n = \sum_{i=1}^{n} a_i$$

The partial sums form a new sequence $\{s_n\}$ which may or may not have a limit. If $\lim_{n\to\infty} s_n = s$ exists, then this limit is called the **sum of the series**.

Definition:

Given a series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$, let s_n denote the nth partial sum.

$$s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \ldots + a_n$$

If the sequence $\{s_n\}$ is convergent and $\lim_{n\to\infty} s_n = s$ exists, then the series $\sum_{n=1}^{\infty} a_n$ is ***convergent***, the number s is called the ***sum of the series***, and we write

$$a_1 + a_2 + a_3 + \ldots = s \quad \text{or} \quad \sum_{n=1}^{\infty} a_n = s$$

If the series is not convergent, we say it is ***divergent***.

The Geometric Series

$$a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1} \qquad a \neq 0$$

Each term is obtained from the preceding one by multiplying it by the common ratio r.
(The example we examined previously is geometric with $a = \frac{2}{3}$ and $r = \frac{1}{3}$.)

When $r = 1$, then $s_n = a + a + \cdots + a = na \to \pm\infty$. So, the series diverges.

When $r \neq 1$, we have

$$s_n = a + ar + ar^2 + \cdots + ar^{n-1}$$

and

$$rs_n = ar + ar^2 + ar^3 + \cdots + ar^n$$

Subtracting these two equations gives us

$$s_n - rs_n = a - ar^n$$

$$s_n = \frac{a(1-r^n)}{1-r}$$

If $-1 < r < 1$, we know from the previous section that $r^n \to 0$ as $n \to \infty$, and so

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{a(1-r^n)}{1-r} = \frac{a}{1-r} - \left(\frac{a}{1-r}\right)\lim_{n\to\infty} r^n = \frac{a}{1-r}$$

If $r \le -1$ or $r > 1$, we know from the previous section that $\{r^n\}$ is divergent, which means the geometric series is divergent also.

<u>Geometric Series</u>:

The geometric series

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots$$

is convergent if $|r| < 1$ and its sum is

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$$

If $|r| \geq 1$, the geometric series is divergent.

If your common ratio *r* does not have an exponent of $n-1$, that's okay. Just put the first term of the series in the numerator. The sum of a geometric series is $\frac{\text{1st term}}{1-r} = \frac{\text{1st term}}{1-\text{common ratio}}$.

<u>Example 3:</u> Find the sum of the geometric series $10 - 4 + \frac{8}{5} - \frac{16}{25} + \cdots$

We'll write it in summation notation and use the $\frac{a}{1-r}$ formula for the sum of a geometric series.

$$\frac{2^1}{5^{-1}} - \frac{2^2}{5^0} + \frac{2^3}{5^1} - \frac{2^4}{5^2} + \cdots = \sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{2^n}{5^{n-2}} \right) = \sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{2^n}{5^n} \right) \left(\frac{1}{5^{-2}} \right)$$

$$= \sum_{n=1}^{\infty} (-1)^n (-1) \left(\frac{2^n}{5^n} \right) \left(\frac{1}{5^{-2}} \right) = -25 \sum_{n=1}^{\infty} \left(-\frac{2}{5} \right)^n$$

$$= -25 \cdot \frac{-\frac{2}{5}}{1 - \left(-\frac{2}{5} \right)} = -25 \cdot \frac{-\frac{2}{5}}{1 + \frac{2}{5}} \cdot \left(\frac{5}{5} \right) = -25 \cdot \frac{-2}{5+2} = \frac{50}{7}$$

<u>Example 4:</u> Is the series $\sum_{n=1}^{\infty} 3^{2n} 11^{1-n}$ convergent or divergent?

$$\sum_{n=1}^{\infty} 3^{2n} 11^{1-n} = \sum_{n=1}^{\infty} \frac{9^n}{11^{n-1}} = \sum_{n=1}^{\infty} 9 \left(\frac{9}{11} \right)^{n-1}$$

Because $|r| = \frac{9}{11} < 1$, the series is convergent. The sum is $\frac{9}{1 - \frac{9}{11}} = \frac{9}{\left(\frac{2}{11} \right)} = \frac{99}{2}$.

Example 5: Write the number $1.27\overline{8}$ as a ratio of integers.

$$1.27\overline{8} = 1.27 + \frac{8}{10^3} + \frac{8}{10^4} + \frac{8}{10^5} + \cdots$$

After the first term, we have a geometric series with $a = \frac{8}{10^3}$ and $r = \frac{1}{10}$.

$$1.27\overline{213} = 1.27 + \frac{\left(\frac{8}{10^3}\right)}{1-\frac{1}{10}} = \frac{127}{100} + \frac{\left(\frac{8}{1000}\right)}{\left(\frac{9}{10}\right)} = \frac{127}{100} + \frac{2}{225} = \frac{1151}{900}$$

Example 6: Show that the series $\sum_{n=1}^{\infty} \frac{2}{4n^2-1}$ converges and find its sum.

Use partial fraction decomposition: $\frac{2}{4n^2-1} = \frac{1}{2n-1} - \frac{1}{2n+1}$

$$\begin{aligned} s_n &= \sum_{n=1}^{n}\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right) \\ &= \left(\frac{1}{1} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{5} - \frac{1}{7}\right) + \left(\frac{1}{7} - \frac{1}{9}\right) + \cdots + \left(\frac{1}{2n-1} - \frac{1}{2n+1}\right) \\ &= 1 - \frac{1}{2n+1} \end{aligned}$$

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty}\left(1 - \frac{1}{2n+1}\right) = 1 - 0 = 1$$

<u>Theorem</u>:

If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$.

<u>NOTE</u>: The converse of this theorem is NOT true! That is, just because $\lim_{n\to\infty} a_n = 0$ does not imply that the series is convergent. (See the harmonic series in Example 7 on page 753.)

<u>The nth-Term Test for Divergence</u> (usually called the <u>Divergence Test</u>):

If $\lim_{n\to\infty} a_n$ does not exist or if $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

Example 7: Show that the series $\sum_{n=1}^{\infty} \frac{3n^2+4n-9}{5n^2}$ is divergent.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{3n^2+4n-9}{5n^2} = \frac{3}{5} \neq 0$$

Since the individual terms converge to a nonzero number, the series diverges from the Divergence Test (*n*th-Term Test).

<u>Theorem</u>:

If $\sum a_n$ and $\sum b_n$ are convergent series, then so are $\sum ca_n$ (c is a constant), $\sum (a_n + b_n)$, and $\sum (a_n - b_n)$, and the sums are:

1) $\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n$
2) $\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$
3) $\sum_{n=1}^{\infty} (a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$

Example 8: Show that the series $\sum_{n=1}^{\infty} \frac{2^n+3^{n+1}}{7^{n-1}}$ converges and find its sum.

$$\sum_{n=1}^{\infty} \frac{2^n+3^{n+1}}{7^{n-1}} = \sum_{n=1}^{\infty} \frac{2^n}{7^{n-1}} + \sum_{n=1}^{\infty} \frac{3^{n+1}}{7^{n-1}}$$
$$= \sum_{n=1}^{\infty} 2\left(\frac{2}{7}\right)^{n-1} + \sum_{n=1}^{\infty} 9\left(\frac{3}{7}\right)^{n-1} = 2\sum_{n=1}^{\infty} \left(\frac{2}{7}\right)^{n-1} + 9\sum_{n=1}^{\infty} \left(\frac{3}{7}\right)^{n-1}$$

Both of the series in the last step are convergent geometric series. Therefore, the series $\sum_{n=1}^{\infty} \frac{2^n+3^{n+1}}{7^{n-1}}$ is convergent.

The sum is $2\cdot\frac{1}{1-\frac{2}{7}} + 9\cdot\frac{1}{1-\frac{3}{7}} = \frac{2}{5/7} + \frac{9}{4/7} = \frac{14}{5} + \frac{63}{4} = \frac{371}{20}$