10.2: Infinite Series

Given a sequence $\{a_n\}_{n=1}^{\infty}$, the sum of the sequence has the form

$$a_1 + a_2 + a_3 + \ldots + a_n + \ldots$$

and is called an **infinite series** (or just simply a **series**). This series is denoted by

$$\sum_{n=1}^{\infty} a_n \quad \text{or} \quad \sum a_n$$

However, does it make sense to have a sum of an *infinite* number of terms?

Example 1: There is no finite sum for the series defined by the sequence $a_n = n^2$

$$1 + 4 + 9 + 16 + 25 + \ldots$$

Example 2: Consider the series below.

$$\frac{2}{3} + \frac{2}{9} + \frac{2}{27} + \frac{2}{81} + \ldots + \frac{2}{3^n} + \ldots$$

If we calculate the sums of the first few terms we obtain

$$\frac{2}{3}, \frac{8}{9}, \frac{26}{27}, \frac{80}{81}, \ldots, 1 - \frac{1}{3^n}, \ldots$$

As we add more terms, the *partial sums* get closer to 1. So, we write

$$\sum_{n=1}^{\infty} \frac{2}{3^n} = 1.$$

Determining whether a series has a finite sum (converges):

We use **partial sums** s_n of the sequence of terms.

$$s_1 = a_1$$
$$s_2 = a_1 + a_2$$
$$s_3 = a_1 + a_2 + a_3$$
$$s_4 = a_1 + a_2 + a_3 + a_4$$
$$\vdots$$
$$s_n = a_1 + a_2 + a_3 + \ldots + a_n = \sum_{i=1}^{n} a_i$$

The partial sums form a new sequence $\{s_n\}$ which may or may not have a limit. If $\lim_{n \to \infty} s_n = s$ exists, then this limit is called the **sum of the series**.

Definition:

Given a series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$, let s_n denote the nth partial sum.

$$s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \ldots + a_n$$

If the sequence $\{s_n\}$ is convergent and $\lim_{n\to\infty} s_n = s$ exists, then the series $\sum_{n=1}^{\infty} a_n$ is ***convergent***, the number s is called the ***sum of the series***, and we write

$$a_1 + a_2 + a_3 + \ldots = s \quad \text{or} \quad \sum_{n=1}^{\infty} a_n = s$$

If the series is not convergent, we say it is ***divergent***.

Example 3: (The Geometric Series)

$$a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1} \qquad a \neq 0$$

Each term is obtained from the preceding one by multiplying it by the common ratio r.
(The example we examined previously is geometric with $a = \frac{2}{3}$ and $r = \frac{1}{3}$.)

Geometric Series:

The geometric series

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots$$

is convergent if $|r| < 1$ and its sum is

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$$

If $|r| \geq 1$, the geometric series is divergent.

Example 4: Find the sum of the geometric series $10 - 4 + \frac{8}{5} - \frac{16}{25} + \cdots$

Example 5: Is the series $\sum_{n=1}^{\infty} 3^{2n} 11^{1-n}$ convergent or divergent? If it converges, what is its sum?

<u>Example 6:</u> Write the number $1.27\overline{8}$ as a ratio of integers.

Telescoping series:

<u>Example 7:</u> Show that the series $\sum_{n=1}^{\infty} \frac{2}{4n^2 - 1}$ converges and find its sum.

<u>Theorem</u>:

If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$.

<u>NOTE</u>: The converse of this theorem is NOT true! That is, just because $\lim_{n\to\infty} a_n = 0$ does not imply that the series is convergent.

<u>The nth-Term Test for Divergence</u> (usually called the <u>Divergence Test</u>):

If $\lim_{n\to\infty} a_n$ does not exist or if $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

<u>Example 8:</u> Show that the series $\sum_{n=1}^{\infty} \frac{3n^2+4n-9}{5n^2}$ is divergent.

<u>Theorem</u>:

If $\sum a_n$ and $\sum b_n$ are convergent series, then so are $\sum ca_n$ (c is a constant), $\sum (a_n + b_n)$, and $\sum (a_n - b_n)$, and the sums are:

1) $\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n$

2) $\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$

3) $\sum_{n=1}^{\infty} (a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$

<u>Example 9:</u> Show that the series $\sum_{n=1}^{\infty} \frac{2^n + 3^{n+1}}{7^{n-1}}$ converges and find its sum.