10.3: The Integral Test

Although we were able to find the sum of geometric series exactly, in general it is difficult to find the sum of a series. In the next several sections, we will develop many tests that will allow us to determine whether a series converges without finding the sum explicitly.

The Integral Test

Suppose f is a continuous, positive, decreasing function on $[1,\infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)\,dx$ is convergent.

That is:

- If $\int_1^{\infty} f(x)\,dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.
- If $\int_1^{\infty} f(x)\,dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.

Note: It is not necessary to start the series of the integral at $n=1$ for this test. For example, to test the series $\sum_{n=4}^{\infty} \frac{1}{(n+5)^3}$ we use $\int_4^{\infty} \frac{1}{(x+5)^3}\,dx$.

Example 1: Test the series $\sum_{n=1}^{\infty} \frac{1}{n^2+2n+5}$ for convergence or divergence.

The function $f(x) = \frac{1}{x^2+2x+5}$ is continuous, positive, and decreasing on $[1,\infty)$, so we can use the Integral Test.

$$\int_1^{\infty} \frac{1}{x^2+2x+5}\,dx = \lim_{t\to\infty} \int_1^{t} \frac{1}{(x+1)^2+2^2}\,dx = \lim_{t\to\infty} \frac{1}{2}\tan^{-1}\left(\frac{x+1}{2}\right)\Bigg]_1^t$$

$$= \lim_{t\to\infty} \frac{1}{2}\left[\tan^{-1}\left(\frac{t+1}{2}\right) - \tan^{-1}\left(\frac{1+1}{2}\right)\right] = \frac{1}{2}\left(\frac{\pi}{2} - \frac{\pi}{4}\right) = \frac{\pi}{8}$$

So, the integral is convergent. Therefore, the series $\sum_{n=1}^{\infty} \frac{1}{n^2+2n+5}$ is convergent.

WARNING: $\sum_{n=1}^{\infty} \frac{1}{n^2+2n+5}$ is NOT equal to $\int_1^{\infty} \frac{1}{x^2+2x+5}\,dx$. Although we know the series converges, the integral does NOT provide the sum of the series.

The *p*-series:

Take note of Example 2 on page 555. The series of the form $\sum_{n=1}^{\infty} \frac{1}{n^p}$ are called *p*-series. The convergence of such a series depends on the value of *p*.

The *p*-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \le 1$.

For example, $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

The harmonic series:

$\sum_{n=1}^{\infty} \frac{1}{n}$ is an important series with a special name. It's called the *harmonic series*.

Example 2: Test the series $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ for convergence or divergence.

$$\int_1^{\infty} \frac{\ln x}{x}\,dx = \lim_{a\to\infty} \int_1^{a} \frac{\ln x}{x}\,dx$$
$$= \lim_{a\to\infty} \frac{(\ln x)^2}{2}\Bigg|_1^a = \lim_{a\to\infty}\left(\frac{(\ln a)^2}{2} - \frac{(\ln 1)^2}{2}\right)$$
$$= \infty - 0 = \infty$$

Since the corresponding improper integral diverges, the series diverges.

Example 3: Test the series $\sum_{n=1}^{\infty} e^{-n}$ for convergence or divergence.

$$\int_1^{\infty} e^{-x}\,dx = \lim_{a\to\infty} \int_1^{a} e^{-x}\,dx$$
$$= \lim_{a\to\infty}\left(-e^{-x}\right)\Bigg|_1^a = \lim_{a\to\infty}\left(-e^{-a} + e^{-1}\right) = \lim_{a\to\infty}\left(-\frac{1}{e^a} + \frac{1}{e}\right)$$
$$= -0 + \frac{1}{e} = \frac{1}{e}$$

Since the corresponding improper integral converges, the series converges.