10.3: The Integral Test

Although we were able to find the sum of geometric series exactly, in general it is difficult to find the sum of a series. In the next several sections, we will develop many tests that will allow us to determine whether a series converges without finding the sum explicitly.

The Integral Test

Suppose *f* is a continuous, positive, decreasing function on $[1,\infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)\,dx$ is convergent.
That is:

- If $\int_1^{\infty} f(x)\,dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.
- If $\int_1^{\infty} f(x)\,dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.

Note: It is not necessary to start the series of the integral at $n=1$ for this test. For example, to test the series $\sum_{n=4}^{\infty} \frac{1}{(n+5)^3}$ we use $\int_4^{\infty} \frac{1}{(x+5)^3}\,dx$.

Example 1: Test the series $\sum_{n=1}^{\infty} \frac{1}{n^2+2n+5}$ for convergence or divergence.

The *p*-series:

Take note of Example 2 on page 761. The series of the form $\sum_{n=1}^{\infty} \frac{1}{n^p}$ are called *p*-series. The convergence of such a series depends on the value of *p*.

The *p*-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leq 1$.

For example, $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

The harmonic series:

$\sum_{n=1}^{\infty} \frac{1}{n}$ is an important series with a special name. It's called the *harmonic series*.

Example 2: Test the series $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ for convergence or divergence.

Example 3: Test the series $\sum_{n=1}^{\infty} e^{-n}$ for convergence or divergence.