10.4: Comparison Tests

In the several comparison tests in this section, the main idea is to compare a given series with another series that is known to be convergent or divergent.

The (Direct) Comparison Test

Suppose $\sum a_n$ and $\sum b_n$ are series with positive terms.

- If $\sum b_n$ is convergent and $a_n \le b_n$ for all *n*, then $\sum a_n$ is also convergent.
- If $\sum b_n$ is divergent and $a_n \ge b_n$ for all *n*, then $\sum a_n$ is also divergent.

Example 1: Test the series $\sum_{n=1}^{\infty} \frac{1}{n^2+2n+5}$ for convergence or divergence.

We found this series was convergent in the last section using the Integral Test. We can also show this with the Direct Comparison Test.

Observe that $\frac{1}{n^2+2n+5} < \frac{1}{n^2}$ for all *n*. Since the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent *p*-series, the series $\sum_{n=1}^{\infty} \frac{1}{n^2+2n+5}$ is also convergent.

Example 2: Test the series $\sum_{n=1}^{\infty} \frac{7}{5n-3}$ for convergence or divergence.

It diverges from the Comparison Test. (Compare to the divergent *p*-series $\sum_{n=1}^{\infty} \frac{7}{5n}$.

<u>Example 3:</u> Test the series $\sum_{n=1}^{\infty} \frac{2}{3^n + 4}$ for convergence or divergence.

It converges by the Comparison Test. Compare to the convergent geometric series $\sum_{n=1}^{\infty} \frac{2}{3^n}$.

NOTE: The terms of the series being tested must be ***smaller*** than the terms of a convergent series, or ***larger*** than the terms of a divergent series. Otherwise, the Direct Comparison Test is useless.

<u>Example 4:</u> Test the series $\sum_{n=1}^{\infty} \frac{3}{4n^3 - 3n}$ for convergence or divergence.

We know $\sum_{n=1}^{\infty} \frac{3}{4n^3} = \frac{3}{4} \sum_{n=1}^{\infty} \frac{1}{n^3}$ converges since $\sum_{n=1}^{\infty} \frac{1}{n^3}$ is a convergent *p*-series. However, with a direct comparison, $\frac{3}{4n^3 - 3n} > \frac{3}{4n^3}$ for all *n*. So, the Direct Comparison Test cannot be used. However, there is another comparison test that will help us.

The Limit Comparison Test

Suppose $\sum a_n$ and $\sum b_n$ are series with positive terms.

- If $\lim_{n \to \infty} \frac{a_n}{b_n} = c$, where c is a finite number and $c > 0$, then either both series converge or both diverge.
- If $\lim_{n \to \infty} \frac{a_n}{b_n} = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.
- If $\lim_{n \to \infty} \frac{a_n}{b_n} = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

Example 5: est the series $\sum_{n=1}^{\infty} \frac{3}{4n^3 - 3n}$ for convergence or divergence.

For the Limit Comparison Test, we will use $a_n = \frac{3}{4n^3 - 3n}$ and $b_n = \frac{3}{4n^3}$.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\left(\frac{3}{4n^3 - 3n}\right)}{\left(\frac{3}{4n^3}\right)} = \lim_{n\to\infty} \left(\frac{3}{4n^3 - 3n}\right)\left(\frac{4n^3}{3}\right) = \lim_{n\to\infty} \frac{4n^3}{4n^3 - 3n} = \lim_{n\to\infty} \frac{4}{4 - \frac{3}{n}} = 1$$

Since this limit exists and $\sum_{n=1}^{\infty} \frac{3}{4n^3} = \frac{3}{4}\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges, the series $\sum_{n=1}^{\infty} \frac{3}{4n^3 - 3n}$ also converges.

Example 6: Test the series $\sum_{n=1}^{\infty} \frac{2n^2 + n + 3}{n^4 + n^3 - n^2}$ for convergence or divergence.

It converges through the Limit Comparison Test. Compare to the convergent *p*-series $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

Example 7: Test the series $\sum_{n=1}^{\infty} \frac{2}{3 + \sqrt{n}}$ for convergence or divergence.

It diverges through the Limit Comparison Test. Compare to the divergent *p*-series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$.