# 10.5: The Ratio and Root Tests

**The Ratio Test and Root Test:**

To test for absolute convergence, the Ratio Test and the Root Test can be very useful. (Absolute and conditional convergence are covered in Section 10.6, which we cover *before* Section 10.5).

**The Ratio Test**
(*Useful when exponentials and/or factorials appear in terms.*)

- If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).
- If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L > 1$ or $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.
- If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$, the Ratio Test is inconclusive; that is, no conclusion can be made about the convergence or divergence of $\sum_{n=1}^{\infty} a_n$.

**Example 1:** Test the series $\sum_{n=1}^{\infty} \frac{3^n}{n!}$ for convergence or divergence.

$$\lim_{n\to\infty}\left|\frac{\frac{3^{n+1}}{(n+1)!}}{\frac{3^n}{n!}}\right| = \lim_{n\to\infty}\left|\left(\frac{3^{n+1}}{(n+1)!}\right)\left(\frac{n!}{3^n}\right)\right| = \lim_{n\to\infty}\left|\frac{3}{n+1}\right| = 0$$

Therefore, the series converges absolutely.

**<u>Example 2:</u>** Test the series $\sum_{n=1}^{\infty}\frac{n^n}{n!}$ for convergence or divergence.

$$\lim_{n\to\infty}\left|\frac{\frac{(n+1)^{n+1}}{(n+1)!}}{\frac{n^n}{n!}}\right|=\lim_{n\to\infty}\left(\frac{(n+1)^{n+1}}{(n+1)!}\right)\left(\frac{n!}{n^n}\right)=\lim_{n\to\infty}\left(\frac{(n+1)^{n+1}}{(n+1)}\right)\left(\frac{1}{n^n}\right)$$

$$=\lim_{n\to\infty}\frac{(n+1)^n}{n^n}=\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n$$

$$=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n=e$$

Therefore, the series $\sum_{n=1}^{\infty}\frac{n^n}{n!}$ diverges.

**The Root Test**
(*Useful when nth powers appear in terms.*)

- If $\lim_{n\to\infty}\sqrt[n]{|a_n|}=L<1$, then the series $\sum_{n=1}^{\infty}a_n$ is absolutely convergent (and therefore convergent).
- If $\lim_{n\to\infty}\sqrt[n]{|a_n|}=L>1$ or $\lim_{n\to\infty}\sqrt[n]{|a_n|}=\infty$, then the series $\sum_{n=1}^{\infty}a_n$ is divergent.
- If $\lim_{n\to\infty}\sqrt[n]{|a_n|}=1$, the Root Test is inconclusive; that is, no conclusion can be made about the convergence or divergence of $\sum_{n=1}^{\infty}a_n$ .

**<u>Example 3:</u>** Test the series $\sum_{n=1}^{\infty}\left(\frac{5n^2-7n}{8n^2+n+3}\right)^n$ for convergence or divergence.

$$\lim_{n\to\infty}\sqrt[n]{\left|\left(\frac{5n^2+7n}{8n^2+n+3}\right)^n\right|}=\lim_{n\to\infty}\sqrt[n]{\left(\frac{5n^2+7n}{8n^2+n+3}\right)^n}=\lim_{n\to\infty}\left(\frac{5n^2+7n}{8n^2+n+3}\right)$$

$$=\lim_{n\to\infty}\left(\frac{5+\frac{7}{n}}{8+\frac{1}{n}+\frac{3}{n^2}}\right)=\frac{5}{8}<1$$

Therefore, the series $\sum_{n=1}^{\infty}\left(\frac{5n^2-7n}{8n^2+n+3}\right)^n$ is absolutely convergent.

**Rearrangement of Terms**

Whether a given series is absolutely convergent has an important ramification. Absolutely convergent series behave like finite sums in that rearranging the order of terms has no effect on the sum.

Oddly, conditionally convergent series can be rearranged to give a different sum.

In fact, if $\sum a_n$ is conditionally convergent and *r* is any real number, then there is a rearrangement of $\sum a_n$ so that the sum is equal to *r*.