10.6: Alternating Series, Absolute and Conditional Convergence

So far, the convergence tests we have seen have applied only to series with positive terms. In the next couple of sections, we will examine tests for series with terms that are not necessarily positive.

An alternating series is a series whose terms are alternately positive and negative.

Example 1: $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} n}{2n-1} = 1 - \frac{2}{3} + \frac{3}{5} - \frac{4}{7} + \frac{5}{9} - \frac{6}{11} + \cdots$

In general, the nth term of an alternating series is of the form

$$a_n = (-1)^{n-1} b_n \quad \text{or} \quad a_n = (-1)^n b_n$$

where $b_n = |a_n|$ is positive.

The Alternating Series Test:

If the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n+1} b_n = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + \cdots$$

satisfies

- $b_n \geq 0$ for all n,
- $b_{n+1} \leq b_n$, for all $n \geq N$ for some integer N
- $\lim_{n\to\infty} b_n = 0$

Example 2: Test the alternating harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ for convergence or divergence.

Observe $\frac{1}{n+1} < \frac{1}{n}$ for all n. So $b_{n+1} \leq b_n$ for all n. Also, $\lim_{n\to\infty} \frac{1}{n} = 0$.

Therefore, the alternating series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ is convergent.

NOTE: The harmonic series $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ is divergent, but the alternating harmonic series is convergent.

Example 3: Test the alternating series $\sum_{n=1}^{\infty} \frac{(-1)^n (5n^2 + 2n)}{2n^2 + 3n + 4}$ for convergence or divergence.

We cannot use the Alternating Series Test, because the 2nd Condition of Alternating Test is not satisfied.

This series diverges from the Divergence Test (*n*th-Term Test). (Section 10.2)

Example 4: Test the alternating series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{4n^2}{3n^3 + 8}$ for convergence or divergence.

This series converges by the Alternating Series Test.

(Use the derivative of the function $f(x) = \frac{4x^2}{3x^3 + 8}$ to show decreasing terms.)

(It's easy to show the limit is 0.)

Example 5: Test the series $\sum_{n=1}^{\infty} \frac{\cos n\pi}{n!}$ for convergence or divergence.

This series converges by the Alternating Series Test.

($\cos n\pi = 1, -1, 1, -1, 1, -1, \ldots$, so we can rewrite it as an alternating series.)

Absolute and conditional convergence:

Given a series $\sum a_n$, we consider the series whose terms are the absolute value of the terms of the original series.

$$\sum_{n=1}^{\infty} |a_n| = |a_1| + |a_2| + |a_3| + \cdots$$

A series $\sum a_n$ is called **absolutely convergent** if the series of absolute values $\sum |a_n|$ converges.

A series $\sum a_n$ is called **conditionally convergent** if it is convergent by not absolutely convergent.

<u>Example 6:</u> The series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3} = -1 + \frac{1}{8} - \frac{1}{27} + \frac{1}{64} - \frac{1}{125} + \cdots$ is absolutely convergent because $\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n^3} \right| = \sum_{n=1}^{\infty} \frac{1}{n^3}$ is a convergent *p*-series.

<u>Example 7:</u> The series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ is conditionally convergent. In the last section, we showed this series converges. But, it is not absolutely convergent because $\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$ is a divergent *p*-series.

Theorem:

If a series $\sum a_n$ is absolutely convergent, then the series is convergent.

Example 8: Test the series $\displaystyle\sum_{n=1}^{\infty}\frac{\sin n}{n^4}=\frac{\sin 1}{1^4}+\frac{\sin 2}{2^4}+\frac{\sin 3}{3^4}+\cdots$ for convergence or divergence.

Although some terms are positive and others are negative, this is not an alternating series. However, the Comparison Test can be used on the absolute values.

$$\sum_{n=1}^{\infty}\left|\frac{\sin n}{n^4}\right|=\sum_{n=1}^{\infty}\frac{|\sin n|}{n^4}$$

Since $|\sin n|\le 1$ for all *n*, we have $\dfrac{|\sin n|}{n^4}\le\dfrac{1}{n^4}$. The *p*-series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^4}$ is convergent. So, by the Comparison Test, $\displaystyle\sum_{n=1}^{\infty}\frac{|\sin n|}{n^4}$ is also convergent, which means the original series is absolutely convergent. Therefore, $\displaystyle\sum_{n=1}^{\infty}\frac{\sin n}{n^4}$ is convergent.

Estimates of sums:

Alternating Series Estimation Theorem:

Suppose the alternating series

$$\sum_{n=1}^{\infty}(-1)^{n+1}b_n=b_1-b_2+b_3-b_4+b_5-b_6+\cdots$$

satisfies

- $b_n\ge 0$ for all *n*,
- $b_{n+1}\le b_n$, for all $n\ge N$ for some integer *N*
- $\lim\limits_{n\to\infty}b_n=0$.
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Then, the *n*th partial sum $s_n=b_1-b_2+\ldots+(-1)^{n+1}b_n$ approximates the sum *L* of the series with an error whose absolute value is less than b_{n+1}, the numerical value of the first unused term. Also, the remainder, $L-s_n$, has the same sign as the first unused term.

Example 9: Estimate the error if we approximate the sum of $\sum_{n=1}^{\infty} \frac{(-1)^n}{3^n}$ by using the 6^{th} partial sum.

Example 10: Estimate the error if we approximate the sum of $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$ by using the 6^{th} partial sum. How many terms would be needed to estimate the sum within 10^{-7}?