10.6: Alternating Series, Absolute and Conditional Convergence

So far, the convergence tests we have seen have applied only to series with positive terms. In the next couple of sections, we will examine tests for series with terms that are not necessarily positive.

An alternating series is a series whose terms are alternately positive and negative.

Example 1: $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}n}{2n-1}=1-\frac{2}{3}+\frac{3}{5}-\frac{4}{7}+\frac{5}{9}-\frac{6}{11}+\cdots$

In general, the *n*th term of an alternating series is of the form

$$a_n=(-1)^{n-1}b_n \quad \text{or} \quad a_n=(-1)^n b_n$$

where $b_n=|a_n|$ is positive.

The Alternating Series Test:

If the alternating series

$$\sum_{n=1}^{\infty}(-1)^{n-1}b_n=b_1-b_2+b_3-b_4+b_5-b_6+\cdots$$

satisfies

- $b_{n+1}\le b_n$, for all n,
- $\lim_{n\to\infty}b_n=0$

then the series is convergent.

Example 2: Test the alternating harmonic series $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots$ for convergence or divergence.

Example 3: Test the alternating series $\sum_{n=1}^{\infty} \frac{(-1)^n (5n^2+2n)}{2n^2+3n+4}$ for convergence or divergence.

Example 4: Test the alternating series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{4n^2}{3n^3+8}$ for convergence or divergence.

Example 5: Test the series $\sum_{n=1}^{\infty} \frac{\cos n\pi}{n!}$ for convergence or divergence.

Absolute and conditional convergence:

Given a series $\sum a_n$, we consider the series whose terms are the absolute value of the terms of the original series.

$$\sum_{n=1}^{\infty} |a_n| = |a_1| + |a_2| + |a_3| + \cdots$$

A series $\sum a_n$ is called **absolutely convergent** if the series of absolute values $\sum |a_n|$ converges.

A series $\sum a_n$ is called **conditionally convergent** if it is convergent by not absolutely convergent.

Example 1: The series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3} = -1 + \frac{1}{8} - \frac{1}{27} + \frac{1}{64} - \frac{1}{125} + \cdots$ is absolutely convergent.

Example 2: The series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ is conditionally convergent.

Theorem:

If a series $\sum a_n$ is absolutely convergent, then the series is convergent.

Example 3: Test the series $\sum_{n=1}^{\infty} \frac{\sin n}{n^4} = \frac{\sin 1}{1^4} + \frac{\sin 2}{2^4} + \frac{\sin 3}{3^4} + \cdots$ for convergence or divergence.

Estimates of sums:

Alternating Series Estimation Theorem:

Suppose the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n+1} b_n = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + \cdots$$

satisfies

- $b_n \geq 0$ for all n,
- $b_{n+1} \leq b_n$, for all $n \geq N$ for some integer N
- $\lim_{n \to \infty} b_n = 0$.
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Then, the nth partial sum $s_n = b_1 - b_2 + ... + (-1)^{n+1} b_n$ approximates the sum L of the series with an error whose absolute value is less than b_{n+1}, the numerical value of the first unused term. Also, the remainder, $L - s_n$, has the same sign as the first unused term.

Example 4: Estimate the error if we approximate the sum of $\sum_{n=1}^{\infty} \frac{(-1)^n}{3^n}$ by using the 6th partial sum.

Example 5: Estimate the error if we approximate the sum of $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$ by using the 6th partial sum. How many terms would be needed to estimate the sum within 10^{-7}?