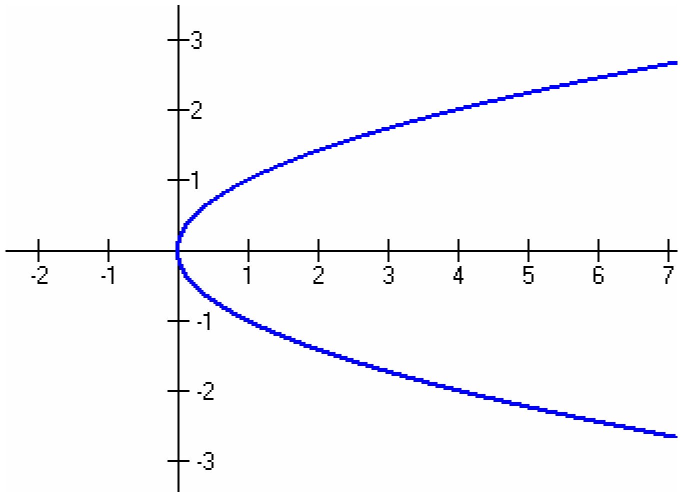
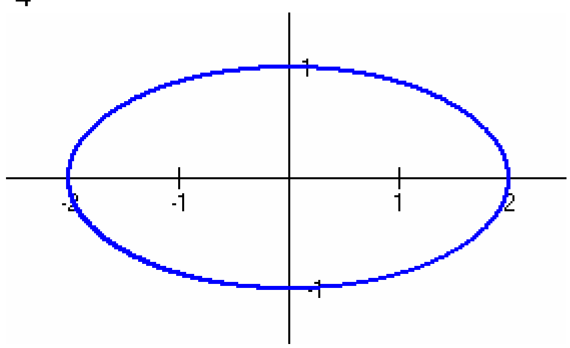
11.1: Parametrizations of Plane Curves

The graph of the equation $x = y^2$ is shown below.



The graph of the equation $\frac{x^2}{4} + y^2 = 1$ is shown below.



Neither of these graphs can be described with functions of the form $y = f(x)$. However, if a third variable *t* is introduced, then both *x* and *y* can be written as functions of *t*.

If both *x* and *y* are given as functions of *t* (*t* is called a parameter), then the equations

$$x = f(t),\ \ y = g(t)$$

are called parametric equations. Each value of *t* determines a point (x, y) in the plane. The curve *C* traced by these points as *t* varies is called a parametric curve.

Although *t* does not always represent time, in many applications it does. We can often interpret the point $(x, y) = (f(t), g(t))$ as the position of a particle at time *t*.

Example 1: Sketch and identify the curve C traced out by the parametric equations $x = t^2$ and $y = t + 1$ for $-2 \leq t \leq 3$. Also, rewrite these equations as a single equation by eliminating the parameter t.

When the parametric equations $x = f(t)$ and $y = g(t)$ are defined for t over an interval, $a \leq t \leq b$, then the point $(f(a), g(a))$ is called the initial point of the curve and the point $(f(b), g(b))$ is called the terminal point of the curve.

Example 2: Sketch the curve for the parametric equations $x = \cos t$ and $y = \sin t$ for $0 \leq t \leq 2\pi$. Also, rewrite these equations as a single equation by eliminating the parameter t.

Example 3: Sketch the curve for the parametric equations $x = \sin 2t$ and $y = \cos 2t$ for $0 \le t \le 2\pi$. Also, rewrite these equations as a single equation by eliminating the parameter *t*.

Example 4: Parametrize the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ three different ways:

a) The ellipse is traversed once counterclockwise.
b) The ellipse is traversed once clockwise.
c) The ellipse is traversed three times clockwise.

Parametrizing a line segment:

A convenient parametrization for the line segment from (a,b) to (c,d) is:

$$x = a + t(c-a)$$
$$y = b + t(d-b)\text{, where } 0 \le t \le 1.$$

Note that when $t = 1$, $(x,y) = (c,d)$.

Example 5: Parametrize the line segment from $(2,-7)$ to $(-3,-4)$.

Example 6: Parametrize the lower half of the parabola $x = y^2 - 5$.

The cycloid:

Suppose a wheel of radius *a* rolls along a horizontal line. The path traced by a point P on the wheel's circumference is called a *cycloid*. The parameter *t* is the angle through which the wheel turns.