11.2: Calculus with Parametric Curves

Tangents:

Even though there is a third variable, the parameter t, in parametric equations, it is still possible to determine the slope of a tangent line, $\frac{dy}{dx}$, to a parametric curve.

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \quad \text{provided } \frac{dx}{dt} \neq 0$$

- A horizontal tangent occurs when $\frac{dy}{dt} = 0$ and $\frac{dx}{dt} \neq 0$.

- A vertical tangent occurs when $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} \neq 0$.

From Calculus 1, we know that not only is the first derivative useful for analyzing graphs, but so is the second derivative.

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

Example 1: Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for the parametric equations $x = 1 + t^2$, $y = t - t^3$.

Example 2: Find the equation of the tangent line to the parametric curve defined by the equations $x = \ln t$, $y = 1 + t^2$ at $t = 1$.

Areas:

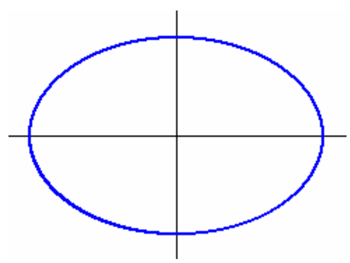
The area under the curve $y = F(x)$ from *a* to *b* is defined as $A = \int_a^b F(x)\,dx$, when $F(x) \geq 0$.

If the curve is defined by parametric equations $x = f(t)$ and $y = g(t)$, and the curve is traversed one time as *t* increases from *α* to *β*, then the area can be defined as follows:

$$A = \int_\alpha^\beta g(t) f'(t)\,dt \quad \text{if } \left(f(\alpha), g(\alpha)\right) \text{ is the leftmost endpoint}$$

$$A = \int_\beta^\alpha g(t) f'(t)\,dt \quad \text{if } \left(f(\beta), g(\beta)\right) \text{ is the leftmost endpoint}$$

Example 3: Find the area of the ellipse defined by $x = a\cos\theta$, $y = b\sin\theta$, $0 \leq \theta \leq 2\pi$. (*a* and *b* are constants.)



Notice the similarity to the area formula for a circle. Since a circle is an ellipse in which the constants a and b in the parametric equations are equal ($r = a = b$), the area for a circle is the familiar $A = \pi r^2$.

Example 4: Find the area under one arch of the cycloid described by the parametric equations $x = t - \sin t$, $y = 1 - \cos t$.

Arc Length:

Suppose that a curve C is described by $x = f(t)$ and $y = g(t)$, $\alpha \le t \le \beta$, where f' and g' are continuous on $[\alpha, \beta]$, and C is traversed exactly one time as t increases from α to β. Then, the length of the curve is given by

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Example 5: Find the length of the curve defined by $x = 3t^2$, $y = 2t^3$, $0 \le t \le 2$.

Surface Area:

If a smooth curve $x = f(t)$, $y = g(t)$, $a \le t \le b$, is traversed exactly once as *t* increases from *a* to *b*, then the areas of the surfaces generated by revolving the curve about the coordinate axes are:

1. Revolution about the *x*-axis ($y \ge 0$)

$$S = \int_a^b 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

2. Revolution about the *y*-axis ($x \ge 0$)

$$S = \int_a^b 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

Example 6: Find the area of the surface formed by rotating the curve defined by $x(t) = 2\cos t$, $y(t) = 3\sin t$, $0 \le t \le \pi$ about the *x*-axis.

Example 7: Find the surface area of a cone of height h and radius r.
Note: Geometry formula is Surface Area= π (radius)(slant height).