11.5: Areas and Lengths in Polar Coordinates

Areas:

Recall the formula for the area of a sector of a circle: $A = \frac{1}{2} r^2 \theta$.
This formula is the basis for the calculation of the area of a region bounded by polar curves.

Let $\Re$ be the region bounded by the polar curve $r = f(\theta)$ and by the rays $\theta = a$ and $\theta = b$, where *f* is a positive continuous function and where $0 < b - a \le 2\pi$. The area of region $\Re$ is given by

$$A = \int_a^b \tfrac{1}{2}\left[f(\theta) \right]^2 d\theta$$

This can also be written as

$$A = \int_a^b \tfrac{1}{2} r^2 d\theta$$

remembering that $r = f(\theta)$.

Example 1: Find the area enclosed by the polar curve $r = 5\sin 3\theta$.

Example 2: Find the area of the region that lies outside the curve $r = 2$ but inside the curve $r = 4\sin\theta$.

Example 3: Find the area of the region inside both $r = 2 + 2\cos\theta$ and $r = 2 - 2\cos\theta$.

Arc Length:

Assuming f' is continuous, the length of the polar curve $r = f(\theta)$ for $a \le \theta \le b$ is

$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

Note: Now we have 4 different representations of the arc length *ds* of a representative section of a curve.

$$ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx \qquad ds = \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy \qquad ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \qquad ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

Example 4: Find the length of the curve $r = 4\cos\theta$, $0 \le \theta \le \pi$.

For many polar curves, it is difficult to calculate the exact value of the length of the curve. In such instances, you may be asked to set up the integral only in certain homework problems. An excellent approximation can then be obtained with a computer or graphing calculator.