6.3: Arc Length

Given a function whose graph consists of connected line segments, finding the length of the graph over an interval $a \le x \le b$ is simply a matter of adding the lengths of the appropriate line segments. However, finding the length of curves requires a little more work. Combining the distance formula with calculus principles results in a formula involving integration.

The Arc Length Formula:

If f' is continuous on $a \le x \le b$, then the length of the curve $y = f(x)$ over $a \le x \le b$ is

$$L = \int_a^b \sqrt{1 + \left[f'(x) \right]^2}\, dx$$

Using Leibniz notation,

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx} \right)^2}\, dx$$

Example 1: Find the length of the arc of $f(x) = \frac{x^3}{6} + \frac{1}{2x}$ on the interval $\frac{1}{2} \le x \le 2$.

Example 2: Find the length of the arc of $y = \ln(\cos x)$ on the interval $0 \le x \le \frac{\pi}{4}$.

If a curve has the equation $x = g(y)$ for $c \le y \le d$ and $g'(y)$ is continuous, then we obtain the following formula by exchanging the roles of *x* and *y* from the previous work.

$$L = \int_c^d \sqrt{1+\left[g'(y)\right]^2}\,dy = \int_c^d \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$$

<u>Example 3:</u> Find the length of the arc of $y^2 = x$ on the interval $0 \le y \le 1$.

<u>Note</u>: Sometimes the arc length integral may be difficult or impossible to evaluate. In those cases, you can use the Trapezoid or Simpson's rule, or use a graphing calculator or computerized algebra system to obtain a numerical approximation.

<u>Example 4:</u> Set up an integral for the length of the arc of $y = \cos x$ over the interval $0 \le x \le \pi$, and use a graphing calculator to estimate the arc length.