6.4: Area of a Surface of Revolution

A surface of revolution is formed when a curve is rotated about a line. This is the lateral boundary of a solid of revolution discussed in Calculus I.

The strategy to calculating the area of a surface of revolution builds upon the process of calculating arc length in the previous section. We approximate the curve with a polygon and its surface rotation will have a similar area to the curve's surface of rotation. The approximating surface will consist of several bands, each of which is formed by the rotation of a line segment about a line. Each band, then, can be thought of as a portion of a circular cone.

General idea:

The surface area of a solid of revolution is given by

$$S = \int_a^b 2\pi r \, ds$$

where r is the radius and ds is the arc length of a representative section.

$$ds = \sqrt{a + \left(\frac{dy}{dx}\right)^2}\, dx \quad \text{or} \quad ds = \sqrt{a + \left(\frac{dx}{dy}\right)^2}\, dy$$

We'll see two more representations of ds later, when we do parametric equations and polar coordinates.

Surface Area

Let *f* be a positive function with a continuous derivative over $a \le x \le b$. The surface area of the surface obtained by rotating the curve $y = f(x)$ about the *x*-axis over $a \le x \le b$ is:

$$S = \int_a^b 2\pi y \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$S = \int_a^b 2\pi f(x) \sqrt{1+\left[f'(x)\right]^2}\, dx$$

If the curve is described as $x = g(y)$ over the interval $c \le y \le d$, then we have:

$$S = \int_c^d 2\pi y \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy$$

If we think of each section of the arc as being ds, the formula can be written in a different way:

Rotation about the *x*-axis:

$$S = \int 2\pi y\, ds \quad .$$

Rotation about the *y*-axis:

$$S = \int 2\pi x\, ds$$

Use either of the two forms below for ds: (choose the one that is the most convenient)

$$ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx \text{ or } ds = \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy$$

Example 1: Find the area of the surface formed by rotating the graph of $f(x) = x^3$ on the interval $0 \le x \le 2$ about the *x*-axis.

Example 2: Find the area of the surface formed by rotating the graph of $x = 1 + 2y^2$ on the interval $1 \le y \le 4$ about the *x*-axis.

Example 3: Find the area of the surface formed by rotating the graph of $f(x) = x^2$ on the interval $0 \le y \le 2$ about the *y*-axis.