8.6: Numerical Integration

Sometimes, it may not be possible to obtain the exact value of a definite integral. This occurs if:

- Finding an antiderivative is difficult or impossible.
- The function to be integrated is not known explicitly, but is determined from instrument readings or collected data.

Approximation of definite integrals is not new. We used Riemann sums to define definite integration. So, Riemann sums could be used as approximate values. However, there are other approximations that are easier to carry out, and more accurate than using rectangles based on left or right endpoints.

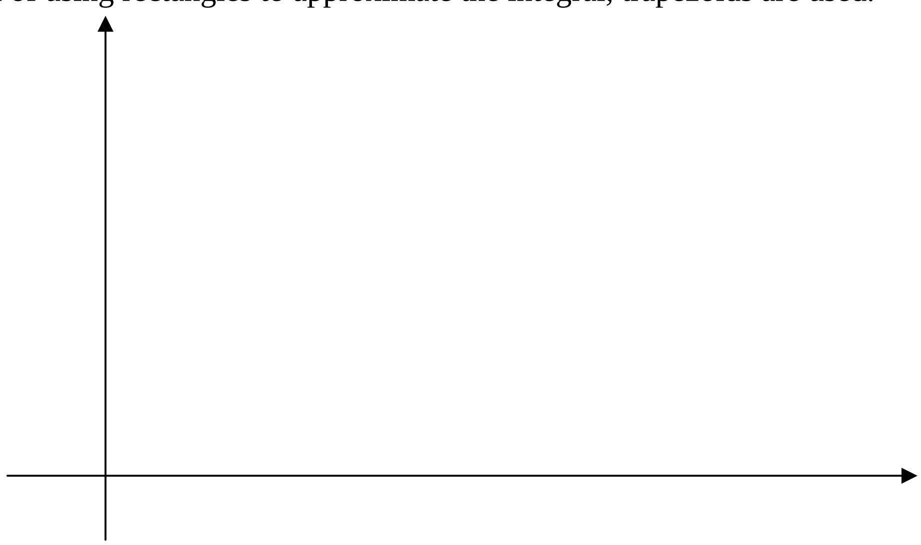
Trapezoid Rule: This averages the left endpoint and right endpoint approximations.

Trapezoid Rule

$$\int_a^b f(x)\,dx \approx T_n = \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n) \right]$$

where $\Delta x = (b-a)/n$ and $x_i = a + i\Delta x$.

Instead of using rectangles to approximate the integral, trapezoids are used.



Example 1: Use the Midpoint Rule and the Trapezoidal Rule to approximate $\int_0^1 4x^3 dx$ with $n = 4$ and $n = 8$.

Midpoint Rule:

Trapezoid Rule:

Error:

The **error** in using an approximation is defined as the amount that should be added to the approximation to make it exact. That is,

$$\int_a^b f(x)\,dx = \text{approximation} + \text{error}$$

Error Bounds:

Suppose $\left|f''(x)\right| \le K$ for $a \le x \le b$. If E_T and E_M are the errors in the Trapezoidal and midpoint rules, then

$$\left|E_T\right| \le \frac{K(b-a)^3}{12n^2} \quad \text{and} \quad \left|E_M\right| \le \frac{K(b-a)^3}{24n^2}$$

Observations:

1. In all methods, more accuracy is obtained by increasing the value of *n*. However, this increases the number of calculations and can lead to round-off error.
2. The errors in the left and right endpoint approximations decrease by approximately a factor of 2 when the value of *n* is doubled.
3. The Midpoint and Trapezoidal rules are much more accurate than the endpoint approximations.
4. The errors in the Trapezoidal and Midpoint Rules decrease by approximately a factor of 4 when the value of *n* is doubled.
5. The size of the error in the Midpoint Rule is about half the size of the error in the Trapezoidal Rule.

Example 2: How large should the value of *n* be in order to guarantee that the Trapezoidal Rule approximation for $\int_0^1 4x^3\,dx$ is accurate to within 0.0001?

Simpson's Rule:

Instead of using straight line segments to approximate a curve, as in the Trapezoidal Rule, Simpson's Rule uses parabolic segments.

Divide $[a,b]$ into n subintervals of equal length $\Delta x = (b-a)/n$, where ***n* is *even***. On each consecutive pair of intervals, there is a unique parabola that can be used to approximate the curve.

Simpson's Rule:

$$\int_a^b f(x)\,dx \approx S_n = \frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right]$$

where n is even and $\Delta x = (b-a)/n$.

Example 3: Use the Simpson's Rule to approximate $\int_0^3 \sqrt{e^x + e^{2x}}\,dx$ with $n = 8$.

Error Bound for Simpson's Rule:

Suppose $\left|f^{(4)}(x)\right| \le K$ for $a \le x \le b$. If E_S is the error in Simpson's Rule, then

$$|E_S| \le \frac{K(b-a)^5}{180n^4}$$

Example 4: How large must we choose n to guarantee that the Simpson's Rule approximation to $\int_1^3 \frac{1}{x^3}\,dx$ is accurate to within 0.001?

Example 5: Use Simpson's Rule to estimate $\int_0^{2.4} f(x)\,dx$ given the measurements shown below.

x	0.0	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0	2.2	2.4
$f(x)$	0.3	0.6	0.4	1.0	1.6	2.3	2.2	2.0	1.6	1.2	0.6	1.1	0.9