8.7: Improper Integrals

In the definition of the definite integral $\int_a^b f(x)\,dx$, we dealt with a finite interval $[a,b]$ and we assumed *f* did not have an infinite discontinuity. We will now extend the definition to include infinite intervals and to include infinite discontinuities. When either situation occurs, the integral is called an **improper integral**.

Type I: Infinite Intervals

- If $f(x)$ is continuous on exists on $[a,\infty)$, then
$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx$$
provided the limit exists.
- If $f(x)$ is continuous on exists on $(-\infty,b]$, then
$$\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty}\int_t^b f(x)\,dx$$
provided the limit exists.

The improper integrals $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^b f(x)\,dx$ are said to be **convergent** if the corresponding limit exists and **divergent** if the limit does not exist.

If both $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^a f(x)\,dx$ are convergent, then
$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx$$
In this situation, any real number *a* may be used.

Example 1: Evaluate $\int_1^\infty \frac{1}{x}\,dx$.

Example 2: Evaluate $\int_1^\infty \frac{1}{x^3}\,dx$.

See pp. 480-481 for proof of the following theorem:

$$\boxed{\int_1^\infty \frac{1}{x^p}\,dx \text{ is convergent if } p > 1 \text{ and divergent if } p \le 1.}$$

Example 3: Evaluate $\int_{-\infty}^{0} 3xe^{-x^2}\,dx$.

Example 4: Evaluate $\int_{-\infty}^{\infty} e^{-x}dx$

Type II: Discontinuous Integrands

- If f is continuous on $[a,b)$ and is discontinuous at b, then
$$\int_a^b f(x)\,dx = \lim_{t\to b^-}\int_a^t f(x)\,dx$$
provided the limit exists.
- If f is continuous on $(a,b]$ and is discontinuous at a, then
$$\int_a^b f(x)\,dx = \lim_{t\to a^+}\int_t^b f(x)\,dx$$
provided the limit exists.

The improper integral $\int_a^b f(x)\,dx$ is said to be **convergent** if the corresponding limit exists and **divergent** if the limit does not exist.

- If f has a discontinuity at c, where $a<c<b$, and both $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ are convergent, then
$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Example 5: Evaluate $\int_3^4 \frac{2}{x-3}\,dx$.

Example 6: Evaluate $\int_0^1 \frac{1}{\sqrt{1-x}}\,dx$.

Example 7: Evaluate $\int_0^5 \frac{1}{x-2}\,dx$.

Be careful! An INCORRECT calculation might be done if the discontinuity is ignored:

$$\int_0^5 \frac{1}{x-2}\,dx = \ln|x-2|\Big]_0^5 = \ln|5-2| - \ln|0-2| = \ln 3 - \ln 2$$

which would seem to indicate the integral converges, which is INCORRECT.

Comparison tests for improper integrals:

Theorem: Direct Comparison Test

Suppose that f and g are continuous functions with $f(x) \ge g(x) \ge 0$ for $x \ge a$.

- If $\int_a^\infty f(x)\,dx$ is convergent, then $\int_a^\infty g(x)\,dx$ is convergent.
- If $\int_a^\infty g(x)\,dx$ is divergent, then $\int_a^\infty f(x)\,dx$ is divergent.

Example 8: Determine whether $\int_0^\infty \frac{1}{x+e^x}\,dx$ converges or diverges.

For $x > 0$, $\frac{1}{x+e^x} < \frac{1}{e^x}$. Also, $\int_0^\infty \frac{1}{e^x}\,dx = \lim_{b\to\infty}\int_0^b e^{-x}\,dx = \lim_{b\to\infty}\left[-e^{-x}\right]_0^b = \lim_{b\to\infty}\left[-e^{-b} + e^{-0}\right] = 0 + 1 = 1$.

Therefore, because $\int_0^\infty \frac{1}{e^x}\,dx$ is convergent, $\int_0^\infty \frac{1}{x+e^x}\,dx$ is convergent also.

Theorem: Limit Comparison Test

Suppose that f and g are continuous and nonnegative on $[a,\infty)$ and that

$$\lim_{x\to\infty}\frac{f(x)}{g(x)} = L\,, \qquad 0 < L < \infty\,.$$

Then,

$$\int_a^\infty f(x)\,dx \quad \text{and} \quad \int_a^\infty g(x)\,dx \quad \text{both converge, or both diverge.}$$

Example 9: Determine whether $\int_4^\infty \frac{x^3+2x}{x^2-1}\,dx$ converges.