7.1: Inverse Functions and their Derivatives

Definition: A function is *one-to-one* (1-1) if each member of the range is associated with exactly one member of the domain.

Phrased another way: A function f is *one-to-one* (1-1) if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$.

Using a graph to determine if a function has an inverse:

Horizontal Line Test: A function is one-to-one (has an inverse) if and only if no horizontal line intersects its graph more than once.

So if you can draw a horizontal line that crosses it twice, then it does not have an inverse.

Example 1:

A one-to-one function has an *inverse* (another function that goes "backwards"). The inverse function reverses whatever the first function did.

The inverse of a function f is denoted by f^{-1}. Read this "*f*-inverse".

Important: The -1 is not an exponent. !!

So $f^{-1}(x) \neq \dfrac{1}{f(x)}$!!!!!

Domain and Range: The range of f is the domain of f^{-1}. The domain of f is the range of f^{-1}.
When you think of inverses, think of exchanging the inputs and outputs, or "switching the *x*'s and *y*'s".

If the function is not one-to-one, you *cannot* do this!!

These two statements mean *exactly* the same thing:

1. f is one-to-one.
2. f has an inverse function.

How to tell if functions are inverses of one another:

Two functions f and g are inverses if:

1. $f(g(x)) = x$ <u>and</u>
2. $g(f(x)) = x$

So, using our notation for f-inverse:

1. $f(f^{-1}(x)) = x$ <u>and</u>
2. $f^{-1}(f(x)) = x$

<u>Example 2:</u> Are $f(x) = 5x + 3$ and $g(x) = \dfrac{x-3}{5}$ inverses?

<u>How to find the inverse of a function</u>: (if it exists!)

1. Replace " $f(x)$ " by "y".
2. Exchange x and y.
3. Solve for y.
4. Replace "y" by " $f^{-1}(x)$ ".
5. Verify!!

Example 3: Find the inverse function of $f(x) = x^3 - 5$.

Example 4: Find the inverse function of $h(x) = \dfrac{4x-2}{6+x}$.

Example 5: Find the inverse function of $f(x) = x^2$.

Graphs of functions and their inverses:

The graphs of f and f^{-1} are symmetric about the line $y = x$.

Example 6:

Calculus of Inverse Functions:

<u>Theorem</u>: If f is a one-to-one continuous function defined on an interval, then its inverse function f^{-1} is also continuous.

<u>Geometric argument</u>: Because the graph of f has "no breaks" and the graph of f^{-1} is the reflection of the graph of f, the graph of f^{-1} would also have "no breaks".

<u>Theorem</u>: If f is a one-to-one differentiable function with inverse function f^{-1} and $f'(f^{-1}(a)) \neq 0$, then the inverse function is differentiable at a and

$$\left(f^{-1}\right)'(a) = \frac{1}{f'\left(f^{-1}(a)\right)}.$$

<u>Proof</u>:

In general, $\left(f^{-1}\right)'(x) = \dfrac{1}{f'\left(f^{-1}(x)\right)}$.

Using Leibniz notation, this is written $\dfrac{dy}{dx} = \dfrac{1}{\left(\dfrac{dx}{dy}\right)}$.

Example 7: Given $f(x) = x^3 - 5$, find $\left(f^{-1}\right)'(3)$.

Example 8: Let $f(x) = x^3 + 5x + 4$, find $\dfrac{df^{-1}}{dx}$ at $x = 10 = f(1)$.