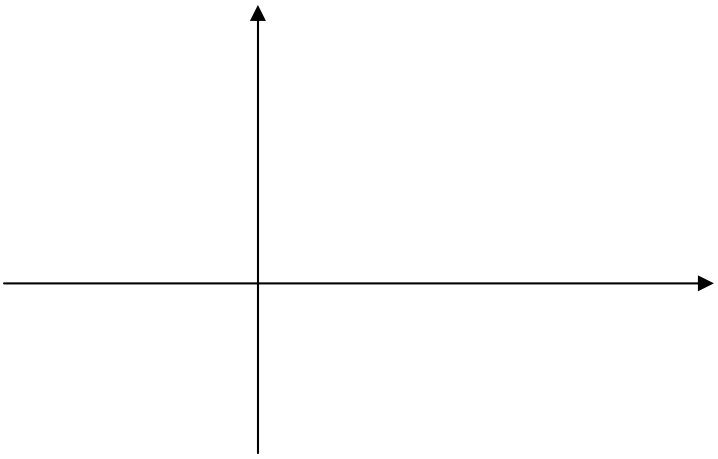
7.2: The Natural Logarithmic Function

The natural logarithm function is defined as

$$\ln x = \int_1^x \frac{1}{t}\,dt, \quad x > 0.$$



For $x > 1$, $\ln x$ can be interpreted as the area under the graph of $y = \frac{1}{t}$ from $t = 1$ to $t = x$.

Note: The integral is not defined for $x < 0$.

For $x = 1$, $\ln x = \int_1^1 \frac{1}{t}\,dt = 0$.

For $x < 1$, $\ln x = \int_1^x \frac{1}{t}\,dt = -\int_x^1 \frac{1}{t}\,dt < 0$.

Recall:

The Fundamental Theorem of Calculus, Part I:

Let f be continuous on the interval $[a,b]$. Then the function g defined by

$$g(x) = \int_a^x f(t)\,dt, \qquad a \le x \le b$$

is continuous on $[a,b]$ and differentiable on (a,b), and $g'(x) = f(x)$.

Apply the Fundamental Theorem of Calculus to the function $f(t)=\frac{1}{t}$.

$$\frac{d}{dx}\left(\int_1^x \frac{1}{t}dt\right)=\frac{1}{x}$$

This means that $\frac{d}{dx}(\ln x)=\frac{1}{x}$.

The Derivative of the Natural Logarithmic Function
$$\frac{d}{dx}(\ln x)=\frac{1}{x}$$

Laws of Logarithms:

If x and y are positive numbers and r is a rational number, then:

1. $\ln(xy)=\ln x+\ln y$
2. $\ln\left(\frac{x}{y}\right)=\ln x-\ln y$
3. $\ln\left(x^r\right)=r\ln x$

Example 1: Expand $\ln\left(\frac{x^3\sqrt{x+5}}{x^2+4}\right)$.

Example 2: Find $\frac{dy}{dx}$ for $y = \ln(2x^5 + 3x)$.

Note: $\frac{d}{dx}\left(\ln u\right) = \frac{1}{u}\frac{du}{dx}$ or, written another way, $\frac{d}{dx}\left(\ln g(x)\right) = \frac{g'(x)}{g(x)}$.

Example 3: Determine $\frac{d}{dx}\left(\ln(\cos x)\right)$.

Example 4: Determine the derivative of $f(x) = \ln\left[\frac{3x^7 + 9x^2}{(x+4)^5}\right]$.

Example 5: Find the derivative of $g(x) = \ln|x|$.

Note that $f(x) = \ln x$ has the same derivative as $g(x) = \ln|x|$.

Therefore $\frac{d}{dx}\ln|x| = \frac{1}{x}$. This means that $f(x) = \ln|x|$ is an antiderivative of $F(x) = \frac{1}{x}$.

$$\int \frac{1}{x}\,dx = \ln|x| + c$$

Recall: The power rule for integrals $\int x^n dx = \frac{x^{n+1}}{n+1}$ had a restriction: $n \neq -1$. Now we can handle this case.

Example 6: Determine $\int \frac{x^2}{x^3+4}\,dx$.

Example 7: Determine $\int \frac{1}{3x}\,dx$.

Example 8: Determine $\int \frac{x^7 - x + 3x^4}{x^5}\,dx$.

Example 9: Find $\int \frac{(\ln x)^4}{x}\,dx$.

Example 10: Find $\int \frac{\ln(3x)}{x}\,dx$.

Example 11: Determine $\int \cot x \, dx$.

Logarithmic differentiation:

To differentiate $y = f(x)$:

1. Take the natural logarithm of both sides.
2. Use the laws of logarithms to expand.
3. Differentiate implicitly with respect to *x*.
4. Solve for y'.

Example 12: Use logarithmic differentiation to find the derivative of

$$y = (x^2 + 2)^5 (2x + 1)^3 (6x - 1)^2 .$$

Example 13: Find y' for $y = \dfrac{(x^3+1)^4 \sin^2 x}{\sqrt[3]{x-2}}$.