7.3: Exponential Functions

Because the natural logarithm function is increasing, it is one-to-one. Thus it has an inverse function which is denoted $\exp x$.

$$\boxed{\exp(x) = y \text{ if and only if } \ln y = x}$$

By the "cancellation equations" for inverse functions:

$$\boxed{\exp(\ln x) = x \text{ and } \ln(\exp x) = x}$$

Also, $\exp(0) = 1$ (because $\ln 1 = 0$) and

$\exp(1) = e$ (because $\ln e = 1$).

Let r be a rational number. According to the third law of logarithms, $\ln(e^r) = r \ln e = r$.

So, by the definition of $\exp x$, we have $\exp(r) = e^r$ for all rational numbers r.

This can be extended to all real numbers.

$$\boxed{\exp(x) = e^x \text{ for } x \in \mathbb{R}}$$

Note: $\mathbb{R}$ is the set of all real numbers.
$\mathbb{Q}$ is the set of all rational numbers.
$\mathbb{N}$ is the set of all natural numbers.
$\mathbb{Z}$ is the set of all integers.
$\mathbb{C}$ is the set of all natural numbers.

The definition of $\exp(x)$ becomes:

$$\boxed{e^x = y \text{ if and only if } \ln y = x.}$$

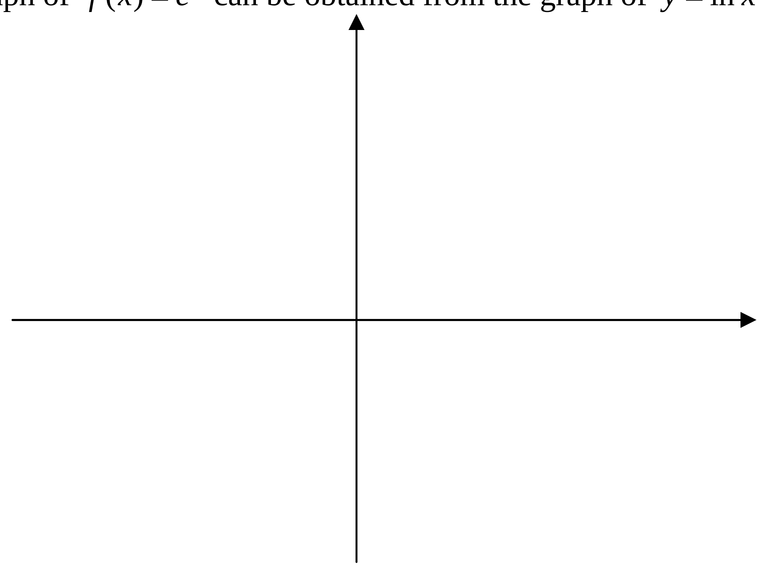
The "cancellation rules" become: $e^{\ln x} = x$ for $x > 0$

$\ln(e^x) = x$ for all x.

Example 1: Solve $\ln x = 3$.

Example 2: Solve $e^{2x+7} = 8$.

The graph of $f(x) = e^x$ can be obtained from the graph of $y = \ln x$.



Properties of the natural exponential function $f(x) = e^x$**:**

Domain:

Range:

Increasing function

Horizontal Asymptote:

$\lim\limits_{x \to -\infty} e^x = 0$ and $\lim\limits_{x \to \infty} e^x = \infty$

Example 3: Find $\lim_{x\to\infty} \dfrac{-2e^{3x}}{e^{3x}-4}$.

Laws of Exponents:

$$e^{x+y} = e^x e^y \qquad e^{x-y} = \frac{e^x}{e^y} \qquad (e^x)^r = e^{rx}$$

Differentiation of the natural exponential function:

Since $y = \ln x$ is differentiable, so is $y = e^x$.

$$\boxed{\frac{d}{dx}(e^x) = e^x}$$

Example 4: Find the derivative of $y = e^{\cos x}$.

Note: $\dfrac{d}{dx}(e^u) = e^u \dfrac{du}{dx}$

Example 5: Find the derivative of $f(x) = 5\sqrt{e^x + 7}$.

Example 6: Find the derivative of $f(x) = e^x \sin x$.

Example 7: Find the derivative of $g(x) = e^{-7x} + 2x^3 - 4e$.

Example 8: Find the derivative of $y = e^{-3x} \tan 2x$

Example 9: Find the derivative of $f(x) = \cos(e^x - x)$.

Example 10: Find the equation of the tangent line to the graph of $f(x) = (e^x + 2)^2$ at the point $(0, 9)$.

Integration of the natural exponential function:

$$\boxed{\int e^x dx = e^x + C}$$

Example 11: Evaluate $\int e^{5t+1} dt$.

Example 12: Evaluate $\int 5x^3 e^{-x^4} dx$

Example 13: Evaluate $\int_0^2 e^{6x} dx$.

Example 14: Determine $\int \frac{e^x}{\sqrt[3]{e^x+1}}\,dx$.

Example 1: $\int \frac{1}{1+e^x}\,dx$

The number e expressed as a limit:

The number e can be defined as

$$e = \lim_{x\to 0}(1+x)^{1/x},$$

or, equivalently,

$$e = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$$

General exponential functions:

For any number $a > 0$, $e^{\ln a} = a$.
So, with any rational number *r* we have: $a^r = \left(e^{\ln a}\right)^r = e^{r \ln a}$.

> (We begin with restricting *r* to rational numbers, but it can be extended to all real numbers).

Therefore, we define exponential functions with base *a* for any real number *x*: $a^x = e^{x \ln a}$.

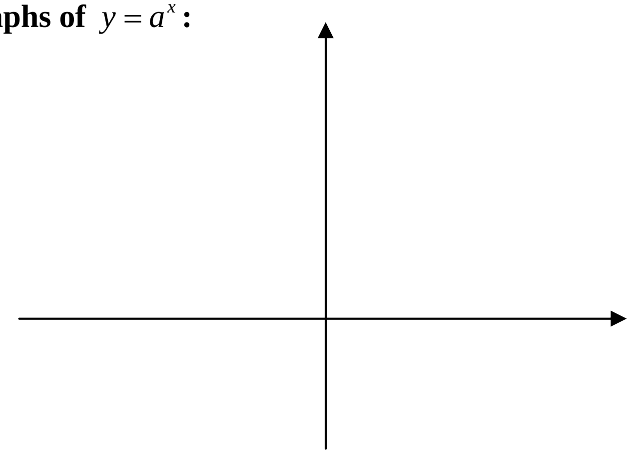
$$\boxed{\underline{\text{The General Exponential Function}}\text{:}\ \ f(x) = a^x}$$

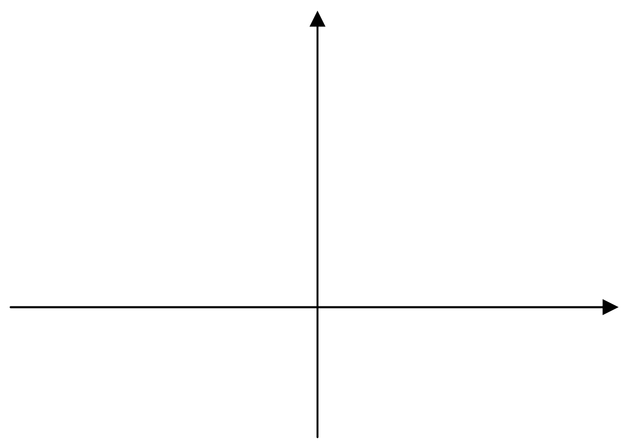
Laws of exponents:

The laws of exponents for e^x can be extended to a^x:

1. $a^{x+y} = a^x a^y$
2. $a^{x-y} = a^x / a^y$
3. $\left(a^x\right)^y = a^{xy}$
4. $(ab)^x = a^x b^x$

Graphs of $y = a^x$:





Differentiation of exponential functions:

$$\boxed{\frac{d}{dx}\left(a^x\right) = a^x \ln a}$$

Proof: $\dfrac{d}{dx}\left(a^x\right) = \dfrac{d}{dx}\left(e^{x \ln a}\right) = e^{x \ln a} \dfrac{d}{dx}\left(x \ln a\right) = a^x \ln a$

Example 2: Find y' for $y = 8^{3x^2+7}$.

Example 3: Differentiate $y = x^{\cos x}$

Integration of general exponential functions:

$$\boxed{\int a^x dx = \frac{a^x}{\ln a} + c, \quad a \neq 1}$$

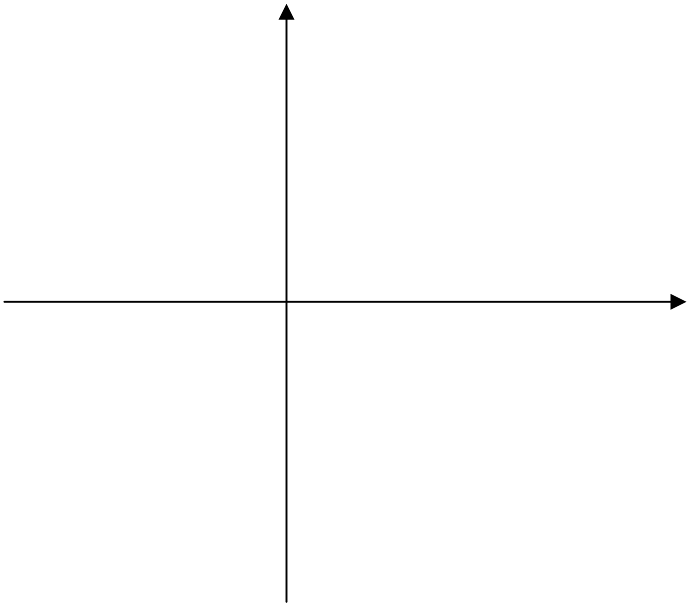
Example 4: Evaluate $\int_0^4 3^x \, dx$.

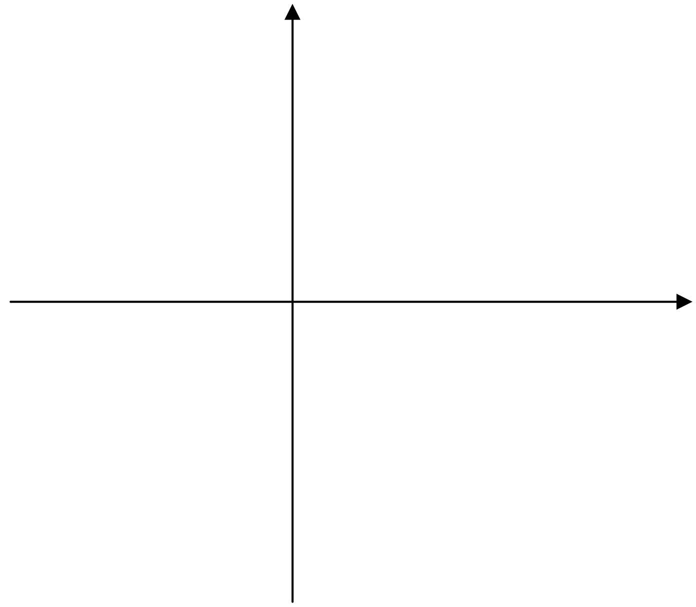
General logarithmic functions:

If $a > 0$ and $a \neq 1$, then $f(x) = a^x$ is a one-to-one function and its inverse is the logarithmic function with base *a*.

$$\boxed{y = \log_a x \quad \Leftrightarrow \quad a^y = x}$$

Graphs of $y = \log_a x$**:**





Change of base formula:

For any positive number a with $a \neq 1$, $\log_a x = \dfrac{\ln x}{\ln a}$.

Differentiation of general logarithmic functions:

$$\boxed{\frac{d}{dx}\left(\log_a x\right) = \frac{1}{x \ln a}}$$

Example 5: Find $\frac{dy}{dx}$ for $y = \log_3(\cos x)$.

Example 6: Find $\frac{dy}{dx}$ for $y = \log_7(2x^3 + \tan x)$.