7.4: Exponential Change and Separable Differential Equations

Differential equations:

A differential equation is an equation that contains an unknown function and one or more of its derivatives. The order of a differential equation is the order of the highest derivative that occurs in the equation.

Example 1: The differential equation $xy' + y = 2x$ is a first order equation.

Example 2: The differential equation $\frac{d^2y}{dt^2} + \frac{dy}{dt} - 6y = 0$ is a second order equation.

A function *f* is a solution of a differential equation if the equation is satisfied when $y = f(x)$ and its derivatives are substituted into the equation.

Example 3: Is $y = 3\sin 2t + \cos 2t$ is a solution of the differential equation $y'' + 4y = 0$?

Solving a differential equation means finding all possible functions that satisfy the equation. You have already seen some very simple differential equations of the form

$$y' = f(x)$$

which requires finding an antiderivative.

Example 4: The general solution of $y' = x^2$ has the form $y = \frac{x^3}{3} + C$.

However, finding the solutions of a differential equation is often difficult, and there is no systematic process that can be used for every differential equation.

For many physical problems which can be modeled by a differential equation, we are often more interested in one specific solution instead of a general family of solutions. An initial-value problem (IVP) is a differential equation that also contains an initial condition of the form $y(t_0) = y_0$. There will be one function among the general solutions that satisfies the differential equation and the initial condition.

Separable differential equations:

While it is generally difficult to find symbolic solutions to many differential equations, there are some varieties of DEs for which certain techniques can be used to find explicit solutions.

A *separable equation* is a first-order differential equation in which the expression for dy/dx can be factored as a function of *x* times a function of *y*. That is,

$$\frac{dy}{dx} = g(x)\,f(y)$$

The term "separable" comes from the fact that the right side of the equation above can be separated into a function of *x* and a function of *y*. Equivalently, assuming $f(y) \neq 0$, we can also write

$$\frac{dy}{dx} = \frac{g(x)}{h(y)}$$

in which $h(y) = 1/f(y)$.

To solve this equation, rewrite it in differential form

$$h(y)\,dy = g(x)\,dx$$

so that all of the "*y*'s" are on the left and all of the "*x*'s" are on the right.
Integrate both sides to obtain

$$\int h(y)\,dy = \int g(x)\,dx$$

This last equation defines *y* implicitly in terms of *x*. In some cases, it is possible to find an explicit form of *y* in terms of *x*.

<u>Example 5:</u> Solve $\dfrac{dy}{dx} = \dfrac{2y}{x-5}$.

<u>Example 6:</u> Solve $xy^2 y' = \ln x$

<u>Example 7:</u> Solve the initial-value problem $y' = x\sin\left(x^2\right)$, $y\left(0\right) = 1$.

Exponential growth and decay:

In many applications, such as the growth of a population or the decay of a radioactive substance, the rate of change of a variable is proportional to the variable itself. This gives the differential equation

$$y' = ky$$

We assume that *y* is positive, which represents a population or a physical substance. When $k > 0$, then the rate of change is positive, and the variable *y* would increase. When $k < 0$, then the rate of change is negative, and the variable *y* would decrease.

This is a *separable* differential equation, which means it can be rearranged so that all the *y*'s are on one side, and the *x*'s are on the other. As we already know, separable differential equations can be solved by integration.

Solve the differential equation $y' = ky$. (Remember, to *solve* means to find y.)

Exponential Growth and Decay

The solution of the initial-value problem

$$\frac{dy}{dt} = ky \qquad y(0) = y_0$$

is

$$y(t) = y_0 e^{kt}$$

Example 8: (Population Growth)
A bacteria culture begins with 800 bacteria and grows at a rate proportional to the culture size. After 4 hours, there are 3000 bacteria.
(a) Find an expression for the number of bacteria after t hours. (Use 5 significant digits for k.)
(b) How many bacteria are there after 7 hours?
(c) How long does it take for the culture to have 40,000 bacteria?

Half-life of a substance:

The *half-life* of a radioactive substance is the amount of time required for half of the radioactive nuclei in a sample to decay. This can be modeled using a differential equation, and is known as exponential decay.

<u>Example 9:</u> The half-life of iron Fe-55 is about 2.7 years. How long will it take for 75% of a sample of iron-55 to disintegrate?

<u>Newton's Law of Cooling (or Warming)</u> states that if the temperature of an object is $T(t)$ at time *t* and T_s is the temperature of its surroundings, then the rate of temperature change follows this differential equation

$$\frac{dH}{dt} = k\left(H - H_s\right)$$

If we let $y(t) = H(t) - H_s$, then $y'(t) = H'(t)$ and the DE becomes

$$\frac{dy}{dt} = ky$$

Thus we have

$$y(t) = y_0 e^{kt}$$
$$H(t) - H_s = \left(H_0 - H_s\right)e^{kt}$$
$$H(t) = \left(H_0 - H_s\right)e^{kt} + H_s$$

where $H_0 - H_s$ is the difference of the initial temperature of the object and the surrounding temperature.

<u>Example 10:</u> (Newton's Law of Cooling)
A temperature of 200°F is often recommended for best results when brewing coffee. Suppose that a freshly brewed cup of coffee has temperature 205°FC, while the room temperature is 70°F. After 5 minutes, the coffee has cooled to 190°F.

For most people, coffee can be drunk comfortably when its temperature is 175°F or less. How long will it take for the coffee to reach this temperature?