7.5: Indeterminate Forms and L'Hospital's Rule

Indeterminate Forms (for limits): $\frac{0}{0}$, $\frac{\infty}{\infty}$, $0\cdot\infty$, $\infty-\infty$, 0^0, ∞^0, 1^∞

Indeterminate forms of the type $\frac{0}{0}$ **and** $\frac{\infty}{\infty}$:

The limit $\lim_{x\to 0}\left(\frac{e^x-1}{2x}\right)$ cannot be evaluated by using the limit laws from Section 2.3. Both the numerator and the denominator tend to 0, so this limit is an "indeterminate form of the type $\frac{0}{0}$." The limit may or may not exist, but it cannot be determined from the current form.

The limit $\lim_{x\to\infty}\left(\frac{\ln x}{x^2}\right)$ also cannot be evaluated by using the usual limit laws. Both the numerator and denominator tend to ∞, so this limit is an "indeterminate form of type $\frac{\infty}{\infty}$."

To evaluate this limit, a new rule is required: L'Hospital's Rule (a.k.a. L'Hôpital's Rule).

L'Hospital's Rule:

Suppose *f* and *g* are differentiable and $g'(x)\neq 0$ near *a* (except possibly at *a*).

Suppose also that $\lim_{x\to a} f(x)=0$ and $\lim_{x\to a} g(x)=0$

or that $\lim_{x\to a} f(x)=\pm\infty$ and $\lim_{x\to a} g(x)=\pm\infty$.

(In other words, there is an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$.)
Then,

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$$

if the limit on the right side exists, or is ∞, or is $-\infty$.

Note: L'Hospital's rule is also valid for $x\to a^+$, $x\to a^-$, $x\to\infty$, and $x\to-\infty$.

Example 1: Find $\lim\limits_{x\to 0}\left(\dfrac{e^x-1}{2x}\right)$.

Example 2: Find $\lim\limits_{x\to\infty}\left(\dfrac{\ln x}{x^2}\right)$.

Example 3: Find $\lim\limits_{x\to\infty}\left(\dfrac{2^x}{x^2}\right)$

Indeterminate forms of the type $0\cdot\infty$:

If $\lim\limits_{x\to a} f(x)=0$ and $\lim\limits_{x\to a} g(x)=\pm\infty$, then it is unclear what $\lim\limits_{x\to a} f(x)g(x)$ is. This limit is an "indeterminate form of the type $0\cdot\infty$." Although this is a limit of a product, it can still be evaluated using L'Hospital's Rule by rewriting the product as a quotient.

$$fg=\frac{f}{1/g} \text{ or } fg=\frac{g}{1/f}.$$

Then the limit will be indeterminate of the form $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$ and L'Hospital's rule can be used.

Example 4: Find $\lim_{x \to -\infty} x^2 e^x$.

Example 5: Find $\lim_{x \to 0^+} \left(\sqrt{x} \ln x \right)$.

Indeterminate forms of the type $\infty - \infty$:

If $\lim_{x \to a} f(x) = \infty$ and $\lim_{x \to a} g(x) = \infty$, then it is unclear what $\lim_{x \to a} \left[f(x) - g(x) \right]$ is. This limit is an "indeterminate form of the type $\infty - \infty$." To use L'Hospital's Rule, we must rewrite the difference as a quotient by finding a common denominator.

Example 6: Find $\lim_{x \to 0} \left(\csc x - \cot x \right)$.

Example 7: Find $\lim_{x\to 1^+}\left(\frac{x}{x-1}-\frac{1}{\ln x}\right)$.

Indeterminate forms of the type 0^0, ∞^0, 1^∞:

If $\lim_{x\to a}\left[f(x)\right]^{g(x)}$ is an indeterminate form of the type 0^0, ∞^0, or 1^∞, then the limit can be handled in one of two ways:

1) Use the natural logarithm. Let $y=\lim_{x\to a}\left[f(x)\right]^{g(x)}$. Then $\ln y = g(x)\cdot\ln f(x)$.

2) Writing the function with the natural exponential $\left[f(x)\right]^{g(x)}=e^{g(x)\cdot\ln f(x)}$.

Example 8: Find $\lim_{x\to 0^+} x^x$.

Example 9: Find $\lim_{x\to 0}\left(e^x + x\right)^{1/\ln x}$.

Example 10: Find $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$.