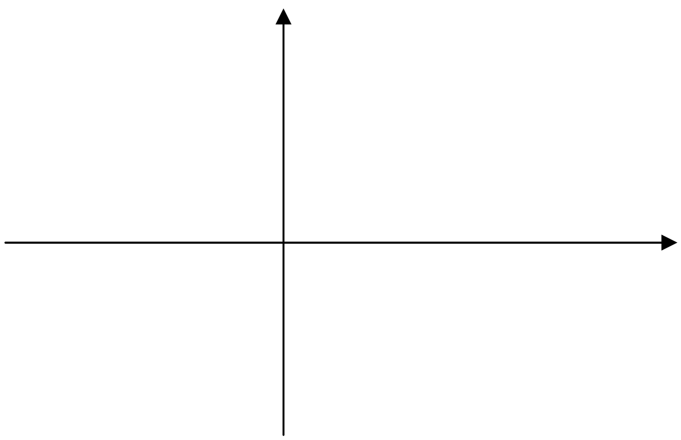
7.6: Inverse Trigonometric Functions

Because none of the trigonometric functions are one-to-one, none of them have an inverse function. To overcome this problem, the domain of each function is restricted so as to produce a one-to-one function.

Inverse sine function:

$$y = \sin^{-1} x \quad \text{if and only if} \quad \sin y = x \text{ and } y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$



Properties of the inverse sine function:

$$\sin^{-1}(\sin x) = x \text{ for } -\frac{\pi}{2} \le x \le \frac{\pi}{2}$$

$$\sin(\sin^{-1} x) = x \text{ for } -1 \le x \le 1$$

Example 1: Evaluate $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$ and $\sin^{-1}\left(-\frac{1}{2}\right)$.

Example 2: Evaluate $\cot\left(\sin^{-1}\frac{3}{7}\right)$

Example 3: Evaluate $\sin\left(\sin^{-1}(-0.54)\right)$.

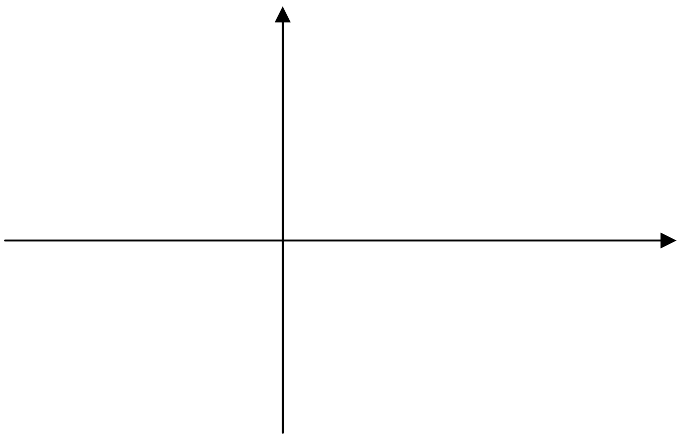
Example 4: Evaluate $\sin\left(\sin^{-1}2\right)$.

Example 5: Evaluate $\sin^{-1}\left(\sin\frac{\pi}{4}\right)$

Example 6: Evaluate $\sin^{-1}\left(\sin\frac{5\pi}{6}\right)$

Inverse cosine function:

$$y = \cos^{-1} x \quad \text{if and only if} \quad \cos y = x \ \text{ and } \ y \in [0, \pi]$$



<u>Properties of the inverse cosine function</u>:

$$\cos^{-1}(\cos x) = x \ \text{ for } \ 0 \le x \le \pi$$
$$\cos(\cos^{-1} x) = x \ \text{ for } \ -1 \le x \le 1$$

<u>Example 7:</u> Evaluate $\cos^{-1}\left(-\frac{1}{2}\right)$

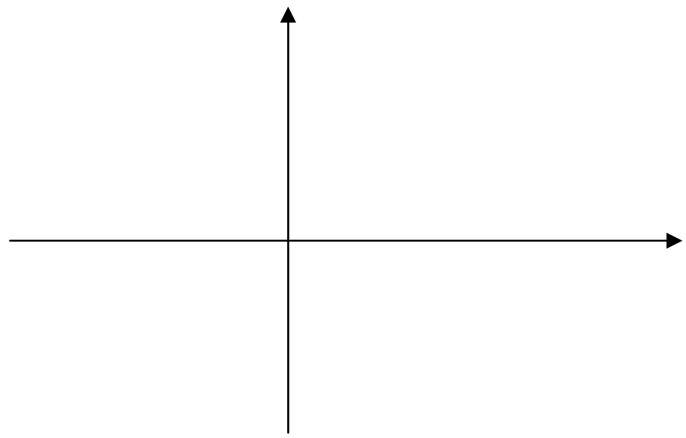
<u>Example 8:</u> Evaluate $\cos^{-1}\left(\cos\left(-\frac{5\pi}{6}\right)\right)$

Inverse tangent function:

$$y = \tan^{-1} x \quad \text{if and only if} \quad \tan y = x \ \text{ and } \ y \in \left(\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$$\lim_{x\to\infty} \tan^{-1} x = \frac{\pi}{2} \ \text{ and } \ \lim_{x\to-\infty} \tan^{-1} x = -\frac{\pi}{2}$$

Graph of $y = \tan^{-1} x$**:**



Inverse secant, cosecant, and cotangent functions:

These are not used as often, and are not defined consistently. Our book defines them as follows:

$$y = \cot^{-1} x \quad \text{if and only if} \quad \cot y = x \ \text{ and } \ y \in (0, \pi)$$

$$y = \csc^{-1} x \quad \text{if and only if} \quad \csc y = x \ \text{ and } \ y \in \left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$$

$$y = \sec^{-1} x \quad \text{if and only if} \quad \sec y = x \ \text{ and } \ y \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$$

An important identity:

$$\cos^{-1} x + \sin^{-1} x = \frac{\pi}{2}$$

Differentiation of the inverse sine function:

Since $y = \sin x$ is continuous and differentiable, so is $y = \sin^{-1} x$.

We want to find its derivative.

$$\boxed{\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}} \quad , -1 < x < 1}$$

<u>**Example 9:**</u> Differentiate $f(x) = \sin^{-1}(2x-7)$.

Differentiation of the inverse cosine function:

$$\boxed{\frac{d}{dx}\left(\cos^{-1} x\right) = -\frac{1}{\sqrt{1-x^2}} \quad , -1 < x < 1}$$

Differentiation of the inverse cosine function:

$$\boxed{\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}}$$

Derivatives of other inverse trigonometric functions:

$\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}}$	$\frac{d}{dx}\left(\csc^{-1} x\right) = -\frac{1}{\lvert x\rvert\sqrt{x^2-1}}$
$\frac{d}{dx}\left(\cos^{-1} x\right) = -\frac{1}{\sqrt{1-x^2}}$	$\frac{d}{dx}\left(\sec^{-1} x\right) = \frac{1}{\lvert x\rvert\sqrt{x^2-1}}$
$\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}$	$\frac{d}{dx}\left(\cot^{-1} x\right) = -\frac{1}{1+x^2}$

Example 10: Find the derivative of $f\left(x\right)=e^{\cot^{-1}2x}$.

Example 11: Find the derivative of $y=\csc^{-1}\left(\tan x\right)$.

Integration formulas involving inverse trigonometric functions:

Two important integration rules come from the inverse trigonometric differentiation rules.

$$\int\frac{1}{\sqrt{1-x^2}}\,dx=\sin^{-1}x+C \qquad \int\frac{1}{x^2+1}\,dx=\tan^{-1}x+C$$

Example 12: Find $\int\frac{1}{\sqrt{1-9x^2}}\,dx$.

Example 13: Find $\int \frac{1}{x^2+7}\,dx$.

More general forms of these integration rules are

$$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C.$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + C$$

Example 14: Find $\int \frac{4x}{x^4+25}\,dx$.

Another antiderivative:

$$\boxed{\int \frac{1}{x\sqrt{x^2-1}}\,dx = \sec^{-1}|x| + C}$$

Example 15: Find $\int \frac{1}{x\sqrt{x^2-4}}\,dx$.

Example 16: Find $\int \frac{1}{4x^2-12x+17}\,dx$.