8.4: Integration of Rational Functions by Partial Fractions

Because

$$\frac{3}{x+1}+\frac{1}{x-2}=\frac{3(x-2)+(x+1)}{(x+1)(x-2)}=\frac{4x-5}{x^2-x-2},$$

we can “reverse” this equation to integrate the rational function on the right side.

$$\int \frac{4x-5}{x^2-x-2}dx=\int\left(\frac{3}{x+1}+\frac{1}{x-2}\right)dx=3\ln|x+1|+\ln|x-2|+C$$

This is an example of integration by partial fractions.

Using Partial Fractions Steps to Integrate a Rational Function $f(x)=\frac{P(x)}{Q(x)}$

1. If the degree of P is greater than or equal to the degree of Q, then divide Q into P by long division to obtain

$$f(x)=S(x)+\frac{R(x)}{Q(x)}$$

2. The denominator $Q(x)$ should be factored as completely into linear factors and irreducible quadratic factors.

3. The rational function should be expressed as a sum of partial fractions of the form

$$\frac{A}{(ax+b)^i} \quad \text{or} \quad \frac{Ax+B}{(ax^2+bx+c)^j}.$$

There are four cases to be considered concerning the denominator $Q(x)$:

Case 1: $Q(x)$ is a product of distinct linear factors.

Case 2: $Q(x)$ is a product of linear factors, some of which are repeated.

Case 3: $Q(x)$ contains irreducible quadratic factors, none of which are repeated.

Case 4: $Q(x)$ contains a repeated irreducible quadratic factor.

Case 1: $Q(x)$ is a product of distinct linear factors.

Example 1: Determine $\int \frac{16}{x^2 - 2x - 15}\,dx$.

<u>Case 2: $Q(x)$ is a product of linear factors, some of which are repeated.</u>

<u>Example 2:</u> Determine $\int \frac{2x^3 - x^2 - 10x - 6}{x^3 - x^2 - 5x - 3}\,dx$.

Case 3: $Q(x)$ contains irreducible quadratic factors, none of which are repeated.

Example 3: Determine $\int \frac{5x^2 - 11x + 14}{x^3 - 3x^2 + 4x - 12}\,dx$.

Case 4: $Q(x)$ contains repeated irreducible quadratic factors.

Example 4: Determine $\int \frac{2x^3 - 5x}{x^4 + 6x^2 + 9}\,dx$.

Rationalizing substitutions:

When an integrand contains an expression such as $\sqrt[n]{g(x)}$, then the substitution $u = \sqrt[n]{g(x)}$ may be appropriate.

Example 5: Determine $\int \frac{1}{x - \sqrt{x+2}}\,dx$.