5.5: Bases Other than *e* and Applications

General Logarithmic and Exponential Functions

For any number $a > 0$, $e^{\ln a} = a$.

Therefore, for any real number *x*, and for any $a > 0$, we have: $a^x = \left(e^{\ln a}\right)^x = e^{x\ln a}$.

W use this to define exponential functions with base *a* for any real number *x*: $a^x = e^{x\ln a}$.

The General Exponential Function: $f(x) = a^x$

Laws of exponents:

1. $a^{x+y} = a^x a^y$
2. $a^{x-y} = a^x / a^y$
3. $\left(a^x\right)^y = a^{xy}$
4. $(ab)^x = a^x b^x$

Differentiation of exponential functions:

$$\boxed{\frac{d}{dx}\left(a^x\right) = a^x \ln a}$$

Proof: $\frac{d}{dx}\left(a^x\right) = \frac{d}{dx}\left(e^{x\ln a}\right) = e^{x\ln a}\frac{d}{dx}\left(x\ln a\right) = a^x \ln a$

Example 1: Find y' for $y = 8^{3x^2+7}$.

<u>Example 2:</u> Find the derivative of $y = 4^{\sin 2x}$.

<u>Example 3:</u> Differentiate $y = x^{\cos x}$

Integration of exponential functions:

$$\boxed{\int a^x dx = \frac{a^x}{\ln a} + c, \quad a \neq 1}$$

<u>Example 4:</u> Evaluate $\int_0^4 3^x \, dx$.

<u>Example 5:</u> Evaluate $\int 5^{2x+7}\,dx$.

General logarithmic functions:

If $a > 0$ and $a \neq 1$, then $f(x) = a^x$ is a one-to-one function and its inverse is the logarithmic function with base a.

$$\boxed{y = \log_a x \quad \Leftrightarrow \quad a^y = x}$$

Change of base formula:

For any positive number a with $a \neq 1$, $\log_a x = \dfrac{\ln x}{\ln a}$.

Differentiation of logarithmic functions:

$$\boxed{\frac{d}{dx}\left(\log_a x\right) = \frac{1}{x \ln a}}$$

Example 6: Find $\frac{dy}{dx}$ for $y = \log_5(6x)$.

Example 7: Find $\frac{dy}{dx}$ for $y = \log_7(2x^3 + \tan x)$.