

1.2: Finding Limits Graphically and Numerically

Limit of a function:

Definition of a Limit:

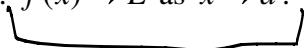
$$\lim_{x \rightarrow a} f(x) = L$$

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The statement above means that we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to a but not equal to a .

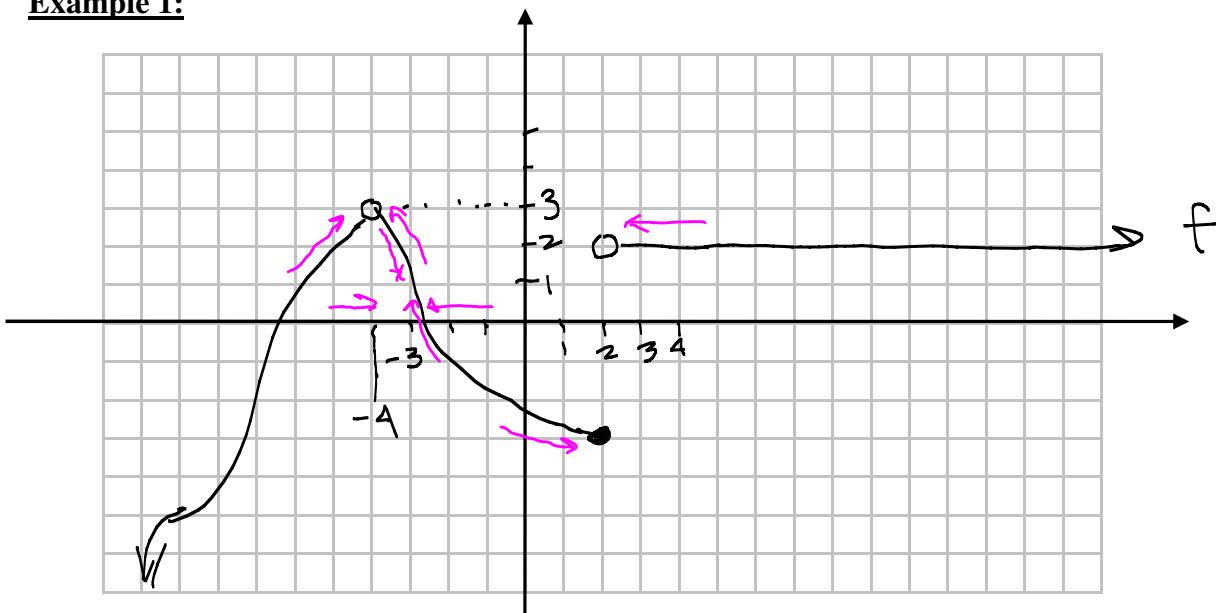
We read this as “the limit of $f(x)$, as x approaches a , is equal to L .”

Alternative notation: $f(x) \rightarrow L$ as $x \rightarrow a$. ($f(x)$ approaches L as x approaches a)



Finding limits from a graph:

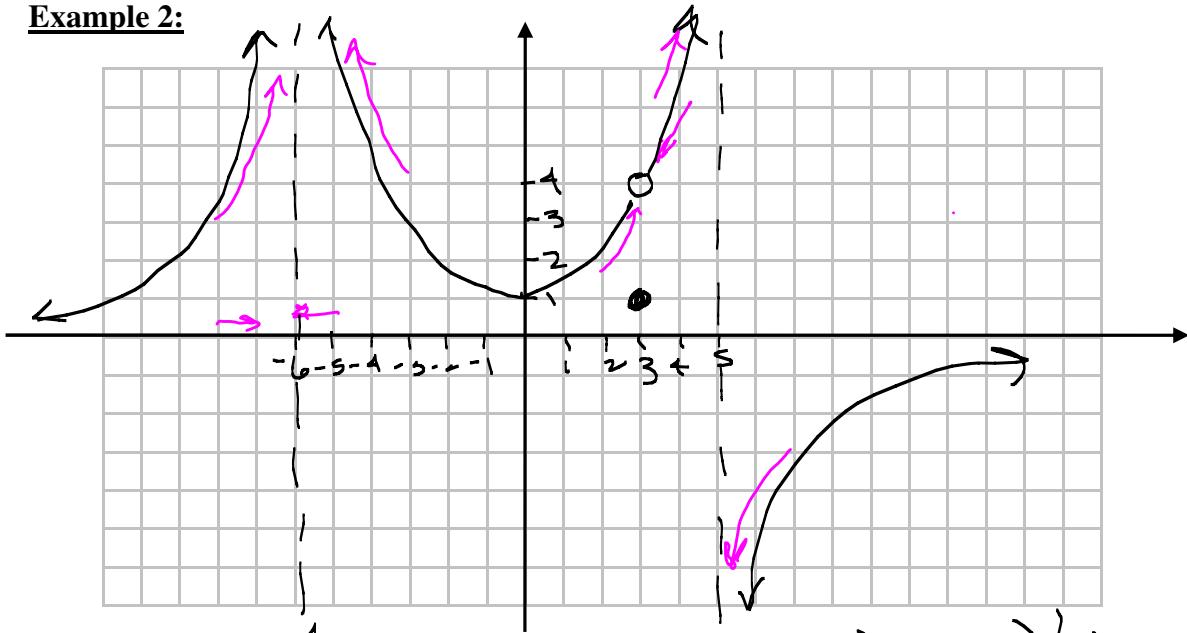
Example 1:



$$\lim_{x \rightarrow -3} f(x) = 1$$

$$\lim_{x \rightarrow -4} f(x) = 3$$

$$\lim_{x \rightarrow 2} f(x) \text{ Does not exist}$$

Example 2:

$$\lim_{x \rightarrow 3} f(x) = 4 \quad (\text{Note that } f(3) = 1)$$

$\lim_{x \rightarrow 5} f(x)$ does not exist.

$$\lim_{x \rightarrow -6} f(x) = \infty$$

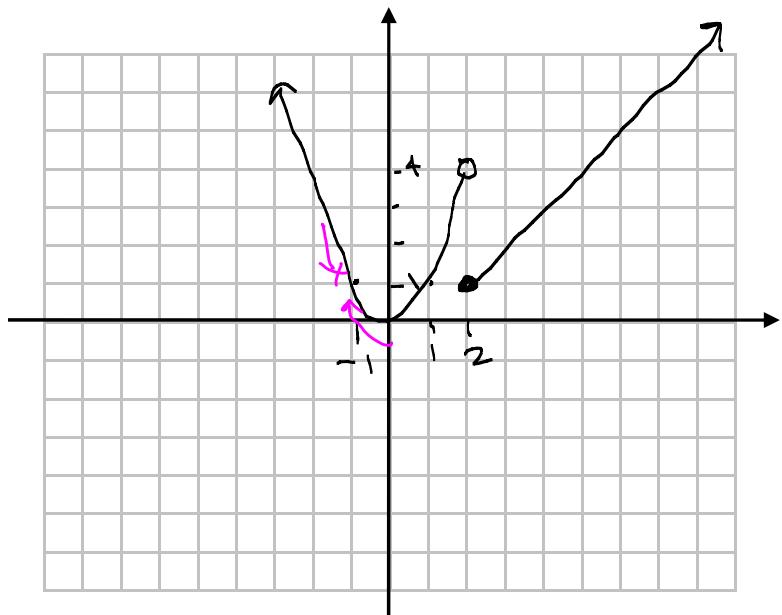
So, $\lim_{x \rightarrow -6} f(x)$ does not exist.

Example 3: Graph the function. Use the graph to determine $\lim_{x \rightarrow 2} f(x)$ and $\lim_{x \rightarrow -1} f(x)$.

$$f(x) = \begin{cases} x-1 & \text{if } x \geq 2 \\ x^2 & \text{if } x < 2 \end{cases}$$

$\lim_{x \rightarrow 2} f(x)$ does not exist

$\lim_{x \rightarrow -1} f(x) = 1$



Finding limits numerically:

Example 4: For the function $f(x) = \frac{x-5}{x^2-25}$, make a table of function values corresponding to values of x near 5.

Use the table to estimate the value of $\lim_{x \rightarrow 5} \frac{x-5}{x^2-25}$.

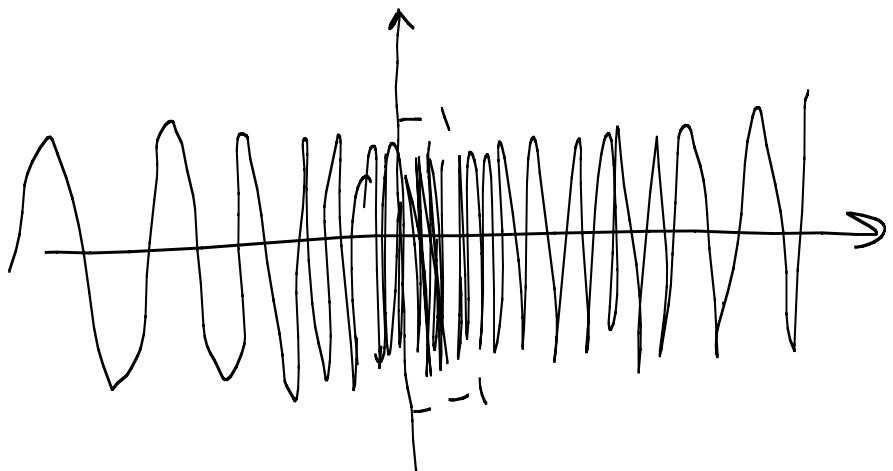
From table, it appears
that $\lim_{x \rightarrow 5} \frac{x-5}{x^2-25} = 0.1$

x	$f(x) = \frac{x-5}{x^2-25}$
4.8	0.10204
4.9	0.10101
4.95	0.1005
4.99	0.1001
4.999	0.10001
5.001	0.09999
5.01	0.0999
5.05	0.0995
5.1	0.09901
5.2	0.09804

Example 5: Make a table of values and use it to estimate $\lim_{x \rightarrow 0} \sin\left(\frac{1}{x}\right)$.

$$\lim_{x \rightarrow 0} \sin\left(\frac{1}{x}\right)$$

This limit
does not
exist.



Common reasons $\lim_{x \rightarrow c} f(x)$ may not exist:

1. $f(x)$ approaches a different value when approached from the left of c , compared to when approached from the right of c .
2. $f(x)$ increases or decreases without bound as x approaches c .
3. $f(x)$ oscillates between two values as x approaches c .

The formal (epsilon-delta) definition of a limit:

Definition:

Let f be a function defined on some open interval that contains the number a , except possibly at a itself. Then

$$\lim_{x \rightarrow a} f(x) = L$$

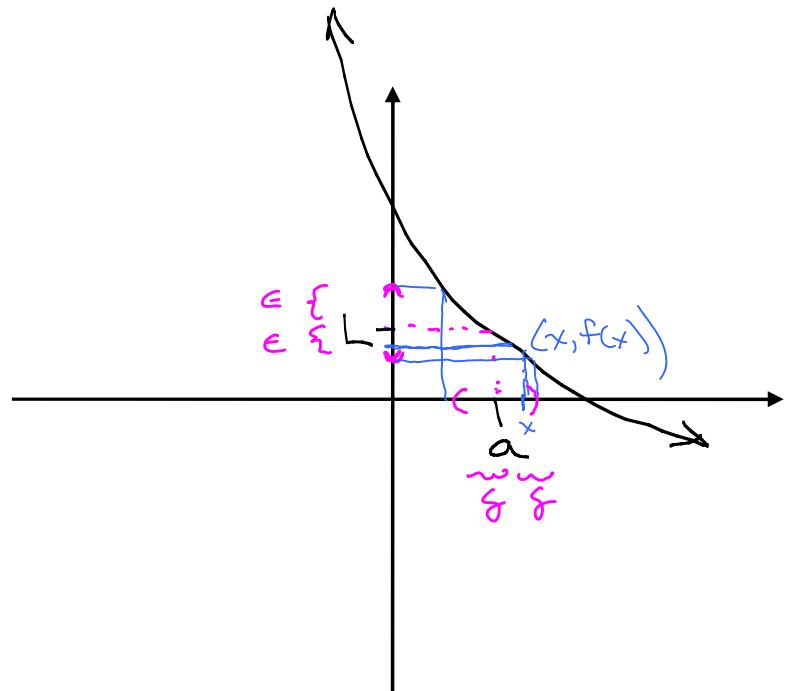
if for every number $\varepsilon > 0$, there is a number $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \text{ whenever } 0 < |x - a| < \delta.$$

Example 6:

Here,

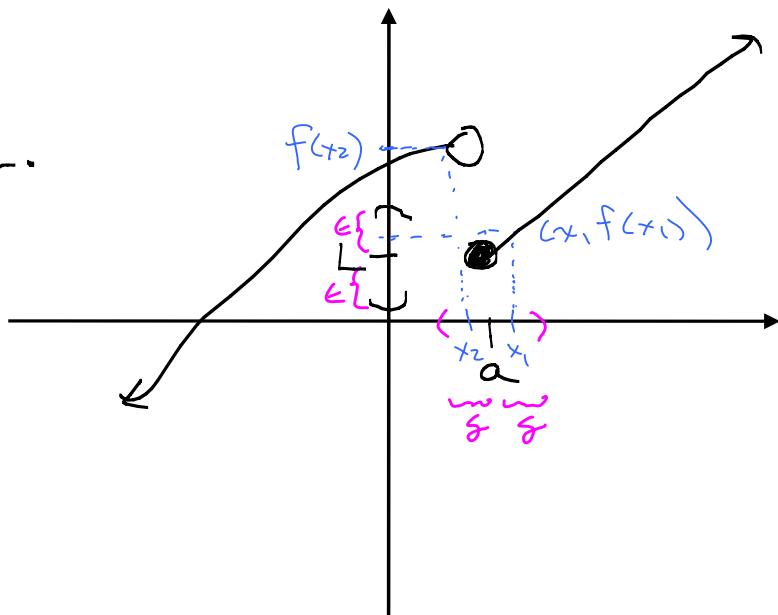
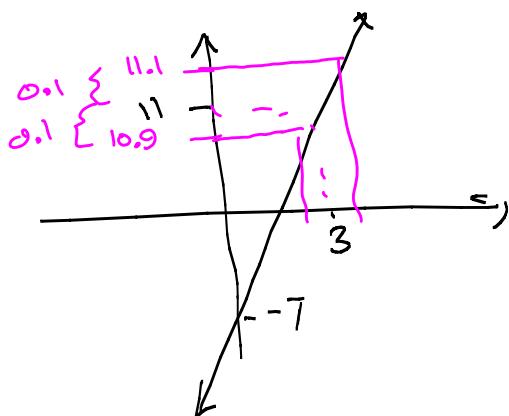
$$\lim_{x \rightarrow a} f(x) = L$$



Example 7:

$f(x_2)$ is not within ϵ of L .

Here, $\lim_{x \rightarrow a} f(x) \neq L$

Example 8: How close to 3 must we take x so that $6x - 7$ is within 0.1 of 11?

$$6x - 7 = 11.1$$

$$6x = 18.1$$

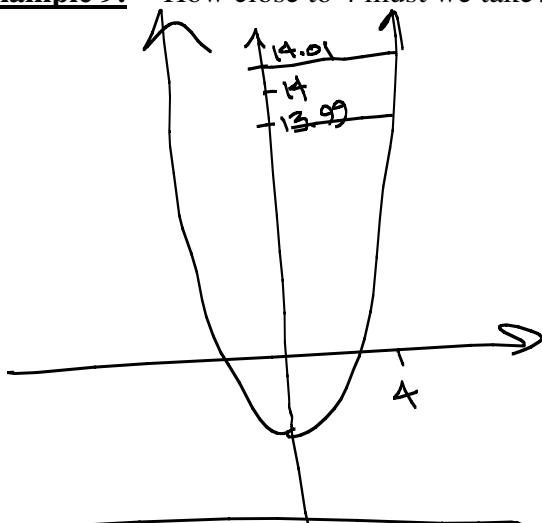
$$x = \frac{18.1}{6} \approx 3.01\bar{6}$$

$$6x - 7 = 10.9$$

$$6x = 17.9$$

$$x = \frac{17.9}{6} \approx 2.98\bar{3}$$

x must be within $0.01\bar{6}$ of 3. (Choose $\delta < 0.01\bar{6}$)

Example 9: How close to 4 must we take x so that $x^2 - 2$ is within 0.01 of 14?

$$x^2 - 2 = 14.01$$

$$x^2 = 16.01$$

$$x \approx 4.001249805$$

$$x^2 - 2 = 13.99$$

$$x^2 = 15.99$$

$$x \approx 3.998749805$$

$$4 - x \approx 0.0012501954$$

Choose $\delta < 0.0012501954$

Example 10: Prove that $\lim_{x \rightarrow 4} (2x - 5) = 3$ using the definition of a limit.

Scratchwork:

Let $\epsilon > 0$.

We want $|(2x - 5) - 3| < \epsilon$.

$$\underbrace{f(x) - L}_{\text{f(x)-L}}$$

$$|2x - 8| < \epsilon$$

$$|2(x - 4)| < \epsilon$$

$$2|x - 4| < \epsilon$$

$$|x - 4| < \frac{\epsilon}{2}$$

$$\text{So, let } \delta = \frac{\epsilon}{2}.$$

Proof:

Let $\epsilon > 0$. Then choose $\delta = \frac{\epsilon}{2}$.

Suppose $0 < |x - 4| < \delta$.

(We now need to show that $|f(x) - L| < \epsilon$)

$$|(2x - 5) - 3| = |2x - 8| = 2|x - 4| < 2\delta = 2\left(\frac{\epsilon}{2}\right)$$

$$= \epsilon.$$



Example 11: Prove that $\lim_{x \rightarrow 3} (5x + 1) = -14$ using the definition of a limit.

Scratchwork:

We want

$$|f(x) - L| = |(5x + 1) - (-14)|$$

$$= |5x + 15|$$

$$= |5(x + 3)|$$

$$= 5|x + 3| < \epsilon$$

$$\underbrace{|x + 3|}_{c} < \frac{\epsilon}{5}$$

$$\left(\text{Same as } \underbrace{|x - c|}_{x-a} < \frac{\epsilon}{5} \right)$$

So, we choose $c = 3$

$$\text{and let } \delta = \frac{\epsilon}{5}.$$

Proof: Let $\epsilon > 0$. Then choose $\delta = \frac{\epsilon}{5}$.

Suppose $0 < |x - 3| < \delta$.

$$\text{Then } |(5x + 1) - (-14)| = |5x + 1 + 14| = |5x + 15|$$

$$= |5(x + 3)| = 5|x + 3| < 5\delta = 5\left(\frac{\epsilon}{5}\right) = \epsilon.$$



Example 12: Prove that $\lim_{x \rightarrow 3} x^2 = 9$ using the definition of a limit.

Scratchwork:

$$\text{We want } |x^2 - 9| < \epsilon$$

$$|(x+3)(x-3)| < \epsilon$$

$$|(x+3)| |x-3| < \epsilon$$

We'll eventually suppose that $|x-3| < \delta$.

Suppose that x is within 1 unit of 3.

$$2 < x < 4$$

$$5 < x+3 < 7$$

$$|x+3| < 7$$

$$\text{Let } c = 7$$

$$\text{Let } \delta = \frac{\epsilon}{7} \quad (\text{or})$$

Example 13: Prove that $\lim_{x \rightarrow 2} (x^2 - x + 6) = 8$ using the definition of a limit.

Scratchwork:

$$|f(x) - L| = |(x^2 - x + 6) - 8|$$

$$= |x^2 - x - 2|$$

$$= |(x-2)(x+1)|$$

$$= |x-2| |x+1|$$

$$\text{so } \delta < c$$

Suppose x within 1 unit of 2.

$$\text{Then } 1 < x < 3$$

$$2 < x+1 < 3$$

$$\text{So } |x+1| < 4$$

$$\text{Let } c = 4.$$

Proof: Let $\epsilon > 0$.

$$\text{Let } \delta = \min\{1, \frac{\epsilon}{7}\}.$$

Suppose $0 < |x-2| < \delta$.

Then $|x-2| < 1$.

$$-1 < x-2 < 1$$

$$1 < x < 3$$

$$5 < x+1 < 7$$

$$|x+1| < 7$$

Correction:

This should be \leq sign, as δ could be either 1 or $\frac{\epsilon}{7}$, depending on which is larger.

$$|x^2 - 9| = |(x+3)(x-3)| = |x+3| |x-3|$$

$$\begin{aligned} & |x-2| < 1 \\ & |x-2| \delta < 1 \left(\frac{\epsilon}{7}\right) \\ & = \epsilon. \end{aligned}$$

Proof: Let $\epsilon > 0$.

$$\text{Choose } \delta = \min\{1, \frac{\epsilon}{4}\}.$$

Suppose $0 < |x-2| < \delta$.

Then $|x-2| < 1$.

$$-1 < x-2 < 1$$

$$1 < x < 3$$

so $2 < x+1 < 4$ and thus $|x+1| < 4$.

$$|(x^2 - x + 6) - 8| = |x^2 - x - 2| = |(x+1)(x-2)|$$

$$= |x+1| |x-2| < 4|x-2| < 4\delta \leq 4\left(\frac{\epsilon}{4}\right) = \epsilon.$$

Important Note: This should be a \leq sign, as x could be either equal to $\frac{\epsilon}{4}$ (if $\frac{\epsilon}{4} \leq 1$) or x could be equal to 1 (thus less than $\frac{\epsilon}{4}$) if $\frac{\epsilon}{4} > 1$.