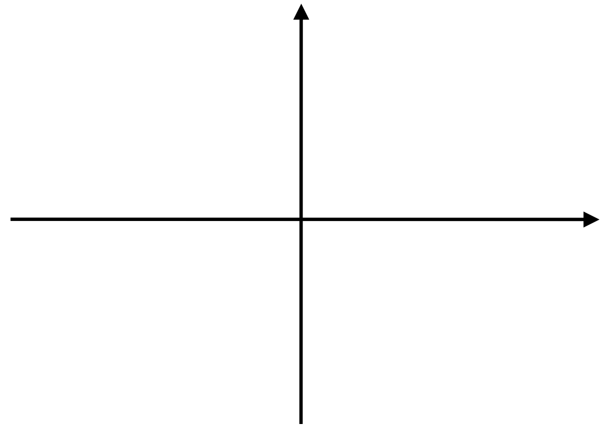
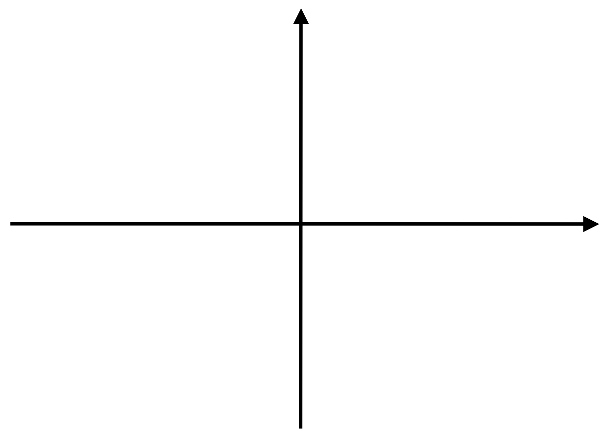
1.3: Evaluating Limits Analytically

Some basic limits:

Example 1: Determine $\lim_{x \to a} x$.



Example 2: Determine $\lim_{x \to a} c$.



Laws (or properties) of limits:

<u>Limit Laws</u>:

Suppose that the limits $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist. Then

1. $\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$

2. $\lim_{x\to a}[f(x)-g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$

3. $\lim_{x\to a}[cf(x)] = c\lim_{x\to a} f(x)$ if c is a constant

4. $\lim_{x\to a}[f(x)g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$

5. $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ if $\lim_{x\to a} g(x) \neq 0$

6. $\lim_{x\to a}[f(x)]^{n/p} = [\lim_{x\to a} f(x)]^{n/p}$, if n and p are integers with no common factor, $p \neq 0$, and provided that $[\lim_{x\to a} f(x)]^{n/p}$ is a real number.

Combining the facts that $\lim_{x\to a} c = c$, $\lim_{x\to a} x = a$, and the limit laws shows that $\lim_{x\to a} x^n = a^n$ where n is a positive integer. This lets us use direct substitution for evaluating limits of polynomials.

<u>Direct Substitution Property</u>:

If f is a polynomial or a rational function and a is in the domain of f, then $\lim_{x\to a} f(x) = f(a)$.

<u>Example 3:</u> Determine $\lim_{x\to 3}(4x^2 - 2x + 1)$.

<u>Example 4:</u> Determine $\lim_{x\to -2} \dfrac{2x^2 - 6x + 5}{x - 3}$.

Example 5: Determine $\lim_{x \to 2} \sqrt[3]{4x - x^4}$.

Limit Law #6 (used in the previous example) is a special case of the following theorem:

Limit of a Composite Function

If f and g are functions such that $\lim_{x \to a} g(x) = L$ and $\lim_{x \to L} f(x) = f(L)$

$$\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right) = f(L)$$

Example 6: Determine $\lim_{x \to 4} \dfrac{x^2 + x - 20}{x^2 - 7x + 12}$.

Example 7: Determine $\lim_{x \to -2} \dfrac{\sqrt{x+3} - 1}{x + 2}$.

Example 8: Determine $\lim_{x \to 3} \dfrac{x+4}{x-3}$.

Limits of trigonometric functions:

These can also be evaluated through direct substitution, thanks to the two limits below, along with the limit laws.

$$\boxed{\lim_{x \to a} \sin x = \sin a \qquad \lim_{x \to a} \cos x = \cos a}$$

Example 9: Determine $\lim_{x \to \frac{\pi}{3}} \tan x$.

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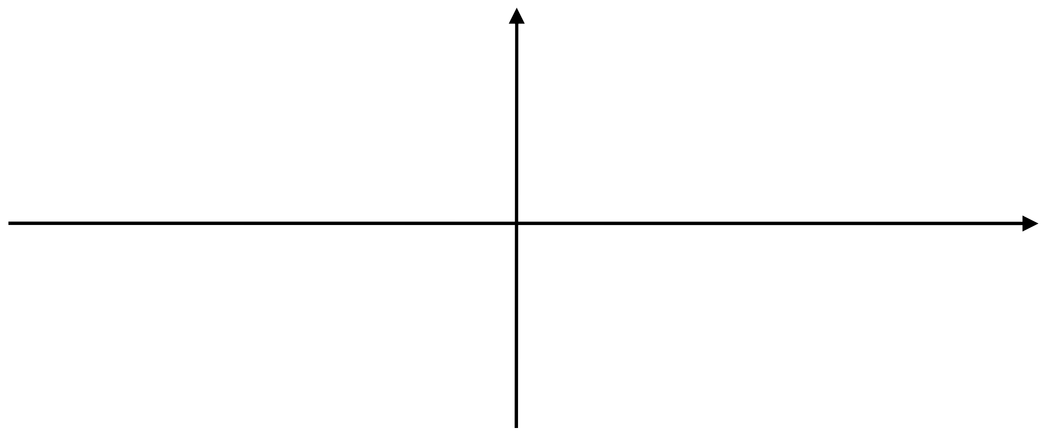
Example 10: Determine $\lim_{x \to \frac{\pi}{4}} \dfrac{\cos x - \sin x}{\cos(2x)}$

<u>The Squeeze (or Sandwich or Pinching) Theorem</u>:

If $g(x) \le f(x) \le h(x)$ for all x in some open interval containing near c, except possibly at c itself. If $\lim_{x \to a} g(x) = \lim_{x \to a} h(x) = L$, then

$$\lim_{x \to a} h(x) = L.$$



<u>Example 11:</u> Use the Squeeze Theorem to show that $\lim_{x \to 0} \frac{\sin x}{x} = 1$.

Two important limits:

$$\lim_{x\to 0}\left(\frac{\sin x}{x}\right)=1$$

$$\lim_{x\to 0}\left(\frac{1-\cos x}{x}\right)=0$$

Note: This means that $\lim_{x\to 0}\left(\frac{x}{\sin x}\right)=1$, $\lim_{x\to 0}\left(\frac{\cos x-1}{x}\right)=0$, and $\lim_{x\to 0}\left(\frac{x}{1-\cos x}\right)$ does not exist.

Example 12: $\lim_{x\to 0}\left(\frac{\cos x\tan x}{x}\right)$

Example 13: $\lim_{x\to \frac{\pi}{2}}\left(\frac{\cos x}{\cot x}\right)$

Example 14: $\lim_{x\to 0}\left(\frac{1-\cos(5x)}{x}\right)$

Example 15: $\lim_{x \to 0} \left(\frac{\sin(3x)}{7x} \right)$

Example 16: $\lim_{x \to 0} \frac{\sin(3x)}{\sin(2x)}$

Example 17: $\lim_{x \to 0} \frac{x^3}{\sin^3(4x)}$

Example 18: $\lim_{x \to 0} \frac{\cot 3x}{\csc 8x}$