

1.3: Evaluating Limits Analytically

Some basic limits:

Example 1: Determine $\lim_{x \rightarrow a} x$.

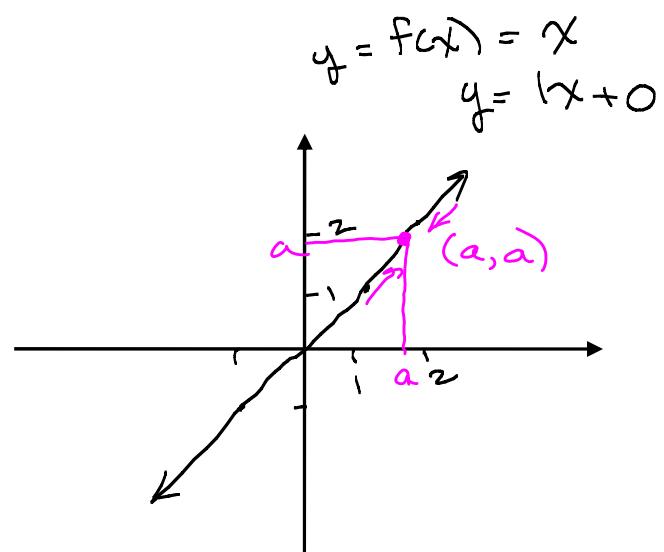
Write as a function: $f(x) = x$

$$\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} x = \boxed{a}$$

x	$f(x) = x$
$a+0.01$	$a+0.01$
$a+0.001$	$a+0.001$
$a+0.0001$	$a+0.0001$

↓

a



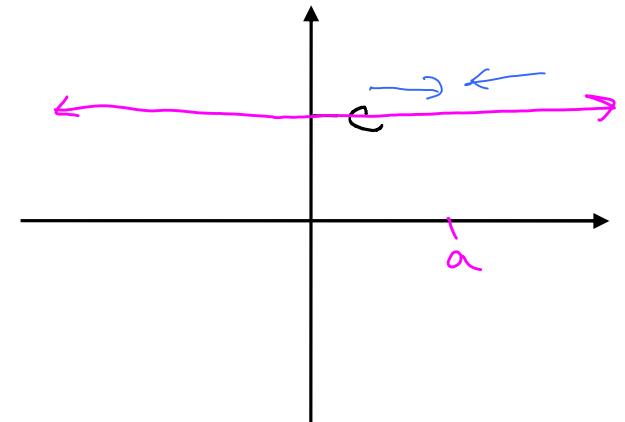
Example 2: Determine $\lim_{x \rightarrow a} c$.

Let $f(x) = c$.

$$\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} c$$

c

$$\begin{array}{c|c} x & f(x) = c \\ \hline a+0.1 & c \\ a+0.01 & c \\ a+0.001 & c \end{array}$$



Laws (or properties) of limits:Limit Laws:

Suppose that the limits $\lim_{x \rightarrow a} f(x)$ and $\lim_{x \rightarrow a} g(x)$ exist. Then

1. $\lim_{x \rightarrow a} [f(x) + g(x)] = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x)$
2. $\lim_{x \rightarrow a} [f(x) - g(x)] = \lim_{x \rightarrow a} f(x) - \lim_{x \rightarrow a} g(x)$
3. $\lim_{x \rightarrow a} [cf(x)] = c \lim_{x \rightarrow a} f(x)$ if c is a constant
4. $\lim_{x \rightarrow a} [f(x)g(x)] = \lim_{x \rightarrow a} f(x) \cdot \lim_{x \rightarrow a} g(x)$
5. $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)}$ if $\lim_{x \rightarrow a} g(x) \neq 0$
6. $\lim_{x \rightarrow a} [f(x)]^{n/p} = [\lim_{x \rightarrow a} f(x)]^{n/p}$, if n and p are integers with no common factor, $p \neq 0$, and provided that $[\lim_{x \rightarrow a} f(x)]^{n/p}$ is a real number.

Combining the facts that $\lim_{x \rightarrow a} c = c$, $\lim_{x \rightarrow a} x = a$, and the limit laws shows that $\lim_{x \rightarrow a} x^n = a^n$ where n is a positive integer. This lets us use direct substitution for evaluating limits of polynomials.

Direct Substitution Property:

If f is a polynomial or a rational function and a is in the domain of f , then $\lim_{x \rightarrow a} f(x) = f(a)$.

Example 3: Determine $\lim_{x \rightarrow 3} (4x^2 - 2x + 1)$.

$$\begin{aligned} & \lim_{x \rightarrow 3} (4x^2 - 2x + 1) \\ &= 4(3)^2 - 2(3) + 1 \\ &= 4(9) - 6 + 1 = 36 - 5 = \boxed{31} \end{aligned}$$

Example 4: Determine $\lim_{x \rightarrow -2} \frac{2x^2 - 6x + 5}{x - 3}$.

$$\begin{aligned} & \lim_{x \rightarrow -2} \frac{2x^2 - 6x + 5}{x - 3} = \frac{2(-2)^2 - 6(-2) + 5}{-2 - 3} = \frac{8 + 12 + 5}{-5} = \frac{25}{-5} = \boxed{-5} \end{aligned}$$

Example 5: Determine $\lim_{x \rightarrow 2} \sqrt[3]{4x - x^4}$.

$$\lim_{x \rightarrow 2} \sqrt[3]{4x - x^4} = \sqrt[3]{4(2) - 2^4} = \sqrt[3]{8 - 16} = \boxed{-2}$$

Limit Law #6 (used in the previous example) is a special case of the following theorem:

Limit of a Composite Function

If f and g are functions such that $\lim_{x \rightarrow a} g(x) = L$ and $\lim_{x \rightarrow L} f(x) = f(L)$

$$\lim_{x \rightarrow a} f(g(x)) = f\left(\lim_{x \rightarrow a} g(x)\right) = f(L)$$

Example 6: Determine $\lim_{x \rightarrow 4} \frac{x^2 + x - 20}{x^2 - 7x + 12}$.

Factor it first:

$$\begin{aligned} \lim_{x \rightarrow 4} \frac{x^2 + x - 20}{x^2 - 7x + 12} &= \lim_{x \rightarrow 4} \frac{(x+5)(x-4)}{(x-4)(x-3)} \\ &= \lim_{x \rightarrow 4} \frac{x+5}{x-3} = \frac{4+5}{4-3} \\ &= \frac{9}{1} = \boxed{9} \end{aligned}$$

Example 7: Determine $\lim_{x \rightarrow -2} \frac{\sqrt{x+3}-1}{x+2}$.

$$\lim_{x \rightarrow -2} \frac{\sqrt{x+3} - 1}{x+2} \left(\frac{\sqrt{x+3} + 1}{\sqrt{x+3} + 1} \right)$$

$$= \lim_{x \rightarrow -2} \frac{(\sqrt{x+3})^2 - (1)^2}{(x+2)(\sqrt{x+3} + 1)} = \lim_{x \rightarrow -2} \frac{x+3-1}{(x+2)(\sqrt{x+3} + 1)}$$

$$= \lim_{x \rightarrow -2} \frac{x+2}{(x+2)(\sqrt{x+3} + 1)} = \lim_{x \rightarrow -2} \frac{1}{\sqrt{x+3} + 1}$$

$$= \frac{1}{\sqrt{-2+3} + 1} = \frac{1}{\sqrt{1} + 1} = \boxed{\frac{1}{2}}$$

Direct Substitution

$$\frac{4^2 + 4 - 20}{4^2 - 7(4) + 12} = \frac{16 - 28 + 12}{16 - 28 + 12} = \frac{0}{0}$$

not defined

This is called an indeterminate form for a limit.

Direct substitution

$$\frac{\sqrt{-2+3} - 1}{-2+2} \Rightarrow \frac{0}{0}$$

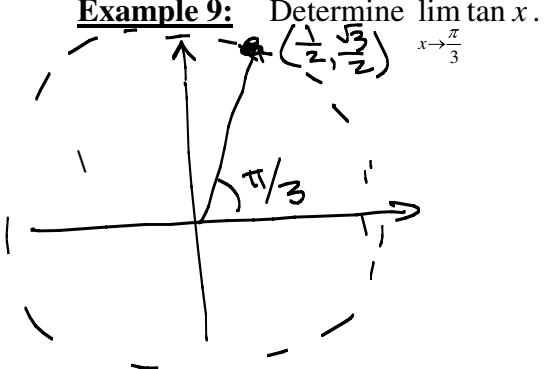
skip for now

Example 8: Determine $\lim_{x \rightarrow 3} \frac{x+4}{x-3}$.

Limits of trigonometric functions:

These can also be evaluated through direct substitution, thanks to the two limits below, along with the limit laws.

$$\lim_{x \rightarrow a} \sin x = \sin a \quad \lim_{x \rightarrow a} \cos x = \cos a$$



$$\lim_{x \rightarrow \frac{\pi}{3}} \tan x = \tan\left(\frac{\pi}{3}\right) = \frac{\sin(\pi/3)}{\cos(\pi/3)} = \frac{\sqrt{3}/2}{1/2} = \frac{\sqrt{3}}{1} = \boxed{\sqrt{3}}$$

Example 10: Determine $\lim_{x \rightarrow \frac{\pi}{4}} \frac{\cos x - \sin x}{\cos(2x)}$

$$\lim_{x \rightarrow \frac{\pi}{4}} \frac{\cos x - \sin x}{\cos 2x} = \lim_{x \rightarrow \frac{\pi}{4}} \frac{\cos x - \sin x}{\cos^2 x - \sin^2 x}$$

$$= \lim_{x \rightarrow \frac{\pi}{4}} \frac{\cos x - \sin x}{(\cos x + \sin x)(\cos x - \sin x)}$$

$$= \lim_{x \rightarrow \frac{\pi}{4}} \frac{1}{\cos x + \sin x} = \frac{1}{\cos \frac{\pi}{4} + \sin \frac{\pi}{4}}$$

$$= \frac{1}{\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}} = \frac{1}{\frac{2\sqrt{2}}{2}} = \boxed{\frac{1}{\sqrt{2}}} + \boxed{\frac{\sqrt{2}}{2}}$$

Direct Sub:

$$\frac{\cos\left(\frac{\pi}{4}\right) - \sin\left(\frac{\pi}{4}\right)}{\cos\left(\frac{2\pi}{4}\right)} = \frac{\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}}{\cos\left(\frac{\pi}{2}\right)}$$

$$\Rightarrow \frac{0}{0}$$

Recall
Double-Angle
Identities

$$\sin 2x = 2 \sin x \cos x$$

$$\cos 2x = \cos^2 x - \sin^2 x$$

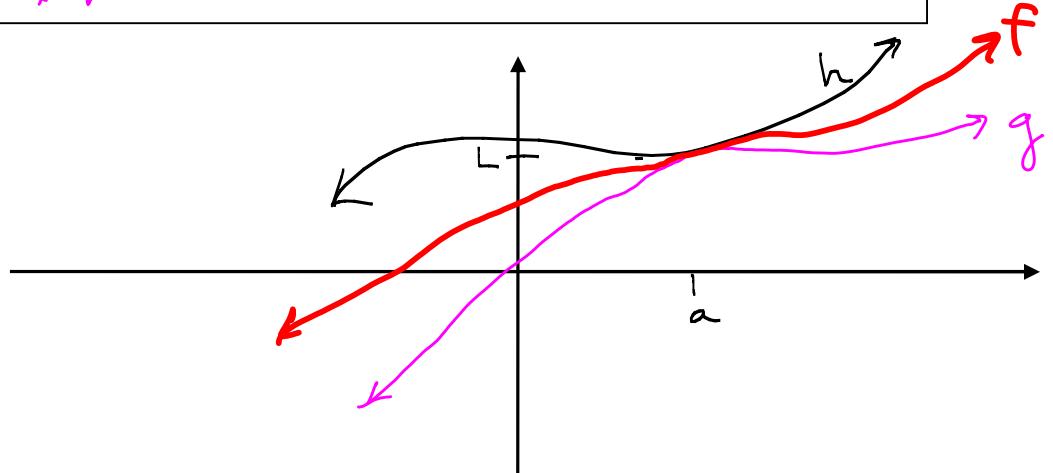
The Squeeze (or Sandwich or Pinching) Theorem:

If $g(x) \leq f(x) \leq h(x)$ for all x in some open interval containing a , except possibly at x itself.

If $\lim_{x \rightarrow a} g(x) = \lim_{x \rightarrow a} h(x) = L$, then

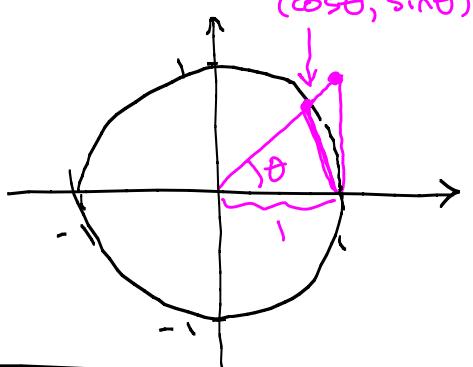
$$\cancel{\lim_{x \rightarrow a} h(x) = L}$$

$$\cancel{\lim_{x \rightarrow a} f(x) = L}$$



Example 11: Use the Squeeze Theorem to show that $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$.

$$\text{Direct sub} \\ \frac{\sin 0}{0} \Rightarrow \frac{0}{0}$$



area of
small Δ \leq area of
circle sector \leq area of
big Δ

sin θ {

$$\frac{1}{2} (\sin \theta)(1) \leq \frac{\theta(1)^2}{2} \leq \frac{1}{2}(1)(\tan \theta)$$

Sector Area

$$\frac{\text{Area sector}}{\pi r^2} = \frac{\theta}{2\pi}$$

$$\begin{aligned} \text{Area sector} &= \frac{\theta}{2\pi} (\pi r^2) \\ &= \frac{\theta r^2}{2} \end{aligned}$$

Multiply by 2:
Divide by $\sin \theta$:

$$\frac{\sin \theta}{2} \leq \frac{\theta}{2} \leq \frac{\tan \theta}{2}$$

$$\sin \theta \leq \theta \leq \tan \theta$$

$$1 \leq \frac{\theta}{\sin \theta} \leq \frac{\tan \theta}{\sin \theta}$$

$$1 \leq \frac{\theta}{\sin \theta} \leq \frac{1}{\cos \theta}$$

$$1 \geq \frac{\sin \theta}{\theta} \geq \cos \theta$$

Take reciprocals
and reverse the
inequalities:

$$\cos \theta \leq \frac{\sin \theta}{\theta} \leq 1$$

So, this holds for negative θ as well.

Two important limits:

Take limits
as $\theta \rightarrow 0$

$$\lim_{\theta \rightarrow 0} \cos \theta = \cos 0 = 1 \quad 1.3.6$$

$$\lim_{\theta \rightarrow 0} 1 = 1$$

What if θ is negative?

$$\begin{aligned} \text{Note that } \frac{\sin(-\theta)}{-\theta} &= -\frac{\sin \theta}{-\theta} \\ &= \frac{\sin \theta}{\theta} \end{aligned}$$

$$\text{Also } \cos(-\theta) = \cos \theta$$

Note: This means that $\lim_{x \rightarrow 0} \left(\frac{x}{\sin x} \right) = 1$, $\lim_{x \rightarrow 0} \left(\frac{\cos x - 1}{x} \right) = 0$, and $\lim_{x \rightarrow 0} \left(\frac{x}{1 - \cos x} \right)$ does not exist.

$$\begin{aligned} \text{Example 12: } \lim_{x \rightarrow 0} \left(\frac{\cos x \tan x}{x} \right) &= \lim_{x \rightarrow 0} \left(\frac{\cos x}{x} \cdot \frac{\tan x}{1} \right) = \lim_{x \rightarrow 0} \left(\frac{\cancel{\cos x} \sin x}{x \cancel{\cos x}} \right) \\ &= \lim_{x \rightarrow 0} \left(\frac{\sin x}{x} \right) = \boxed{1} \end{aligned}$$

$$\begin{aligned} \text{Example 13: } \lim_{x \rightarrow \frac{\pi}{2}} \left(\frac{\cos x}{\cot x} \right) &= \lim_{x \rightarrow \frac{\pi}{2}} \left(\frac{\cos x}{\frac{\cos x}{\sin x}} \right) = \lim_{x \rightarrow \frac{\pi}{2}} \left(\frac{1}{\frac{1}{\sin x}} \right) \\ &= \lim_{x \rightarrow \frac{\pi}{2}} (\sin x) = \sin \frac{\pi}{2} = \boxed{1} \end{aligned}$$

$$\text{Example 14: } \lim_{x \rightarrow 0} \left(\frac{1 - \cos(5x)}{x} \right)$$

Note: as $x \rightarrow 0$, $5x \rightarrow 0$ also.
(because $\lim_{x \rightarrow 0} (5x) = 5(0) = 0$)

$$\begin{aligned} \lim_{x \rightarrow 0} \left(\frac{1 - \cos(5x)}{x} \right) &= \lim_{x \rightarrow 0} \left(\frac{1 - \cos 5x}{x} \right) \left(\frac{5}{5} \right) \\ &= \lim_{x \rightarrow 0} 5 \left(\frac{1 - \cos 5x}{5x} \right) = 5 \lim_{5x \rightarrow 0} \left(\frac{1 - \cos 5x}{5x} \right) \\ &= 5(0) = \boxed{0} \end{aligned}$$

$$\begin{aligned}
 \text{Example 15: } \lim_{x \rightarrow 0} \left(\frac{\sin(3x)}{7x} \right) &= \frac{1}{7} \lim_{x \rightarrow 0} \left(\frac{\sin(3x)}{x} \right) \\
 &= \frac{1}{7} \lim_{x \rightarrow 0} \left(\frac{\sin(3x)}{x} \right) \left(\frac{3}{3} \right) = \frac{3}{7} \lim_{x \rightarrow 0} \left(\frac{\sin(3x)}{3x} \right) \\
 &= \frac{3}{7} \lim_{3x \rightarrow 0} \left(\frac{\sin(3x)}{3x} \right) = \frac{3}{7} (1) = \boxed{\frac{3}{7}}
 \end{aligned}$$

$$\begin{aligned}
 \text{Example 16: } \lim_{x \rightarrow 0} \frac{\sin(3x)}{\sin(2x)} &= \lim_{x \rightarrow 0} \left[\left(\frac{\sin(3x)}{\sin(2x)} \right) \left(\frac{x}{x} \right) \right] \\
 &= \lim_{x \rightarrow 0} \left[\left(\frac{\sin 3x}{x} \right) \left(\frac{x}{\sin 2x} \right) \right] = \left[\lim_{x \rightarrow 0} \left(\frac{\sin 3x}{x} \right) \right] \left[\lim_{x \rightarrow 0} \left(\frac{x}{\sin 2x} \right) \right] \\
 &= \left[\lim_{3x \rightarrow 0} \left(\frac{\sin 3x}{x} \right) \left(\frac{3}{3} \right) \right] \left[\lim_{2x \rightarrow 0} \left(\frac{x}{\sin 2x} \right) \left(\frac{2}{2} \right) \right] \\
 &= \left[3 \lim_{3x \rightarrow 0} \left(\frac{\sin 3x}{3x} \right) \right] \left[\frac{1}{2} \lim_{2x \rightarrow 0} \left(\frac{2x}{\sin 2x} \right) \right] \\
 &= [3(1)] [\frac{1}{2}(1)] = 3(\frac{1}{2}) = \boxed{\frac{3}{2}}
 \end{aligned}$$

$$\begin{aligned}
 \text{Example 17: } \lim_{x \rightarrow 0} \frac{x^3}{\sin^3(4x)} &= \lim_{x \rightarrow 0} \left(\frac{x}{\sin(4x)} \right)^3 \\
 &= \lim_{x \rightarrow 0} \left(\frac{x}{\sin(4x)} \cdot \frac{4}{4} \right)^3 = \lim_{x \rightarrow 0} \left(\frac{4x}{4 \sin 4x} \right)^3 = \frac{1}{4^3} \lim_{x \rightarrow 0} \left(\frac{4x}{\sin 4x} \right)^3 \\
 &= \frac{1}{64} \left(\lim_{4x \rightarrow 0} \left(\frac{4x}{\sin 4x} \right)^3 \right) = \frac{1}{64} (1)^3 = \boxed{\frac{1}{64}}
 \end{aligned}$$

$$\text{Example 18: } \lim_{x \rightarrow 0} \frac{\cot 3x}{\csc 8x}$$

$$\begin{aligned}
 \lim_{x \rightarrow 0} \frac{\cot 3x}{\csc 8x} &= \lim_{x \rightarrow 0} \left(\frac{\cos 3x}{\sin 3x} \right) \left(\frac{1}{\csc 8x} \right) = \lim_{x \rightarrow 0} \left[\left(\frac{\cos 3x}{1} \right) \left(\frac{1}{\sin 3x} \right) \left(\frac{\sin 8x}{1} \right) \right] \\
 &= \lim_{x \rightarrow 0} \left[\left(\frac{\cos 3x}{1} \right) \left(\frac{1}{\sin 3x} \right) \left(\frac{\sin 8x}{1} \right) \left(\frac{x}{x} \right) \left(\frac{3}{3} \right) \left(\frac{8}{8} \right) \right] \\
 &= \frac{8}{3} \lim_{x \rightarrow 0} \left[\left(\frac{\cos 3x}{1} \right) \left(\frac{3x}{\sin 3x} \right) \left(\frac{\sin 8x}{8x} \right) \right] = \frac{8}{3} \left[\lim_{x \rightarrow 0} (\cos 3x) \right] \left[\lim_{3x \rightarrow 0} \left(\frac{3x}{\sin 3x} \right) \right] \left[\lim_{8x \rightarrow 0} \left(\frac{\sin 8x}{8x} \right) \right] \\
 &= \frac{8}{3} [\cos(3(0))] (1)(1) = \frac{8}{3} (1)(1)(1) = \boxed{\frac{8}{3}}
 \end{aligned}$$