

1.4: Continuity and One-Sided Limits

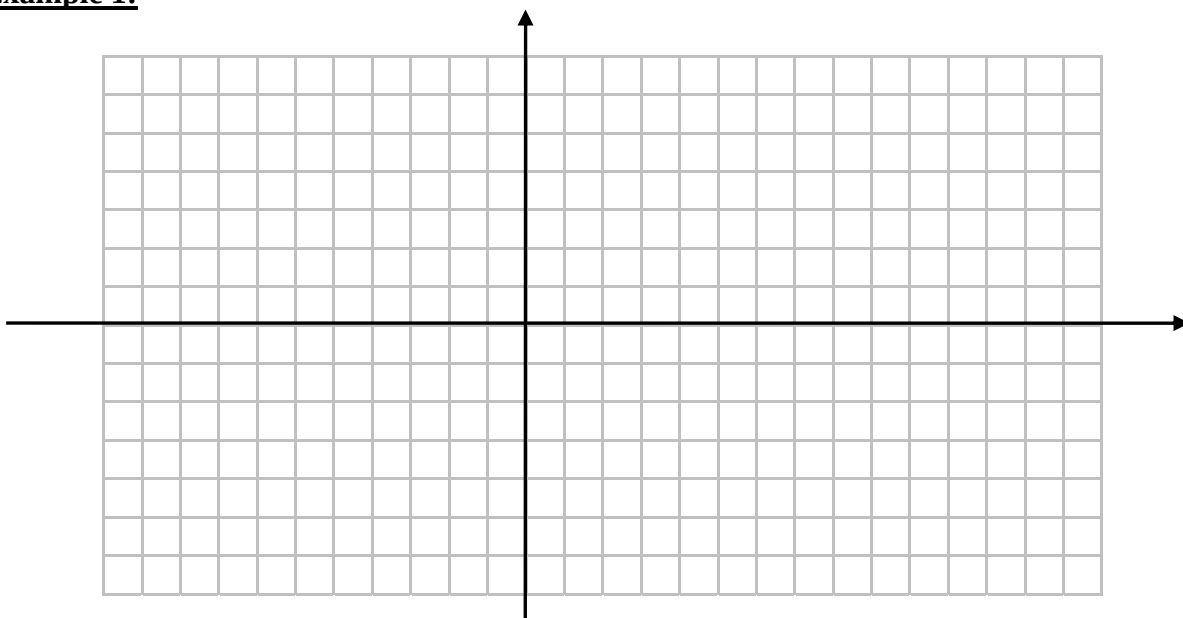
One-Sided Limits:

$\lim_{x \rightarrow a^-} f(x) = L$ means that $f(x)$ approaches L as x approaches a from the left.

$\lim_{x \rightarrow a^+} f(x) = L$ means that $f(x)$ approaches L as x approaches a from the right.

$\lim_{x \rightarrow a} f(x) = L$ if and only if $\lim_{x \rightarrow a^-} f(x) = L$ and $\lim_{x \rightarrow a^+} f(x) = L$.

Example 1:



Example 2: Determine $\lim_{x \rightarrow 1^-} f(x)$, $\lim_{x \rightarrow 1^+} f(x)$, and $\lim_{x \rightarrow 1} f(x)$.

$$f(x) = \begin{cases} x^2 & \text{if } x \leq 1 \\ x-3 & \text{if } x > 1 \end{cases}.$$

Example 3: Determine $\lim_{x \rightarrow -2^-} f(x)$, $\lim_{x \rightarrow -2^+} f(x)$, and $\lim_{x \rightarrow -2} f(x)$.

$$f(x) = \begin{cases} 1-x^3 & \text{if } x \leq -2 \\ 7-x & \text{if } x > -2 \end{cases}.$$

Example 4: Determine $\lim_{x \rightarrow 0} |x|$, if it exists.

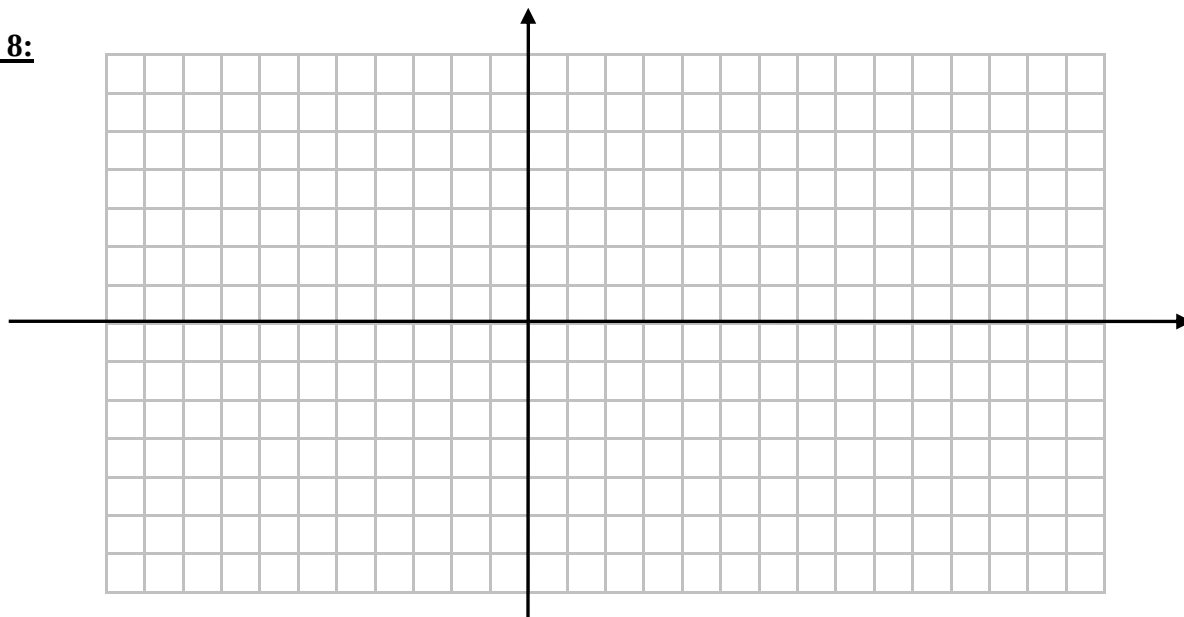
Example 5: Determine $\lim_{x \rightarrow 1} f(x)$ and $\lim_{x \rightarrow -2} f(x)$, where $f(x) = \begin{cases} \sqrt{5-x} & \text{if } x \geq 1 \\ 3x^2 - 7 & \text{if } x < 1 \end{cases}$.

Example 6: Determine $\lim_{x \rightarrow 0} f(x)$, where $f(x) = \begin{cases} \cos x & \text{if } x < 0 \\ 2 & \text{if } x = 0 \\ \sqrt{x+1} & \text{if } x > 0 \end{cases}$.

Example 7: Determine $\lim_{x \rightarrow 1} \frac{|x-1|}{x-1}$.

Continuity of a function:

In most cases, we can think of a continuous function as one that can be drawn “without lifting your pencil from the paper”. In other words, there are no holes, breaks, or jumps.

Example 8:

Definition: A function f is continuous at a number a , an interior point of its domain, if

$$\lim_{x \rightarrow a} f(x) = f(a)$$

Conditions for Continuity:

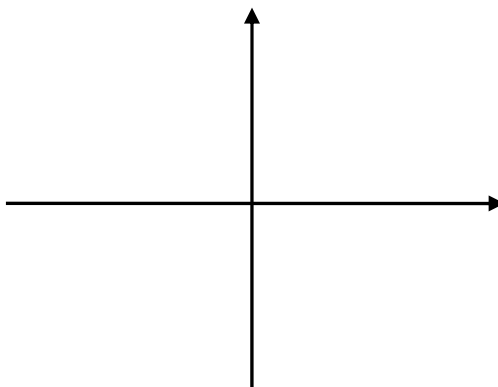
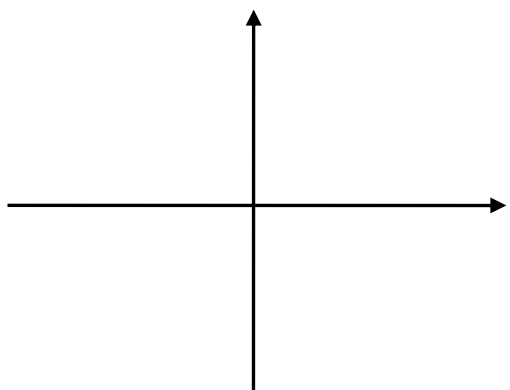
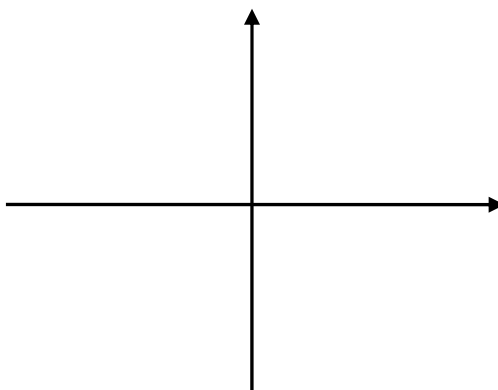
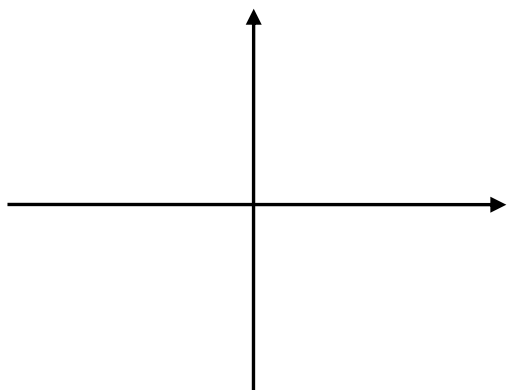
In order for f to be continuous at a , all the following conditions must hold:

1. $f(a)$ is defined.
2. $\lim_{x \rightarrow a} f(x)$ exists.
3. $\lim_{x \rightarrow a} f(x) = f(a)$.

Types of discontinuities:

1. Removable discontinuity
2. Infinite discontinuity
3. Jump discontinuity
4. Oscillating discontinuity

What do these look like? For each type of discontinuity, what condition of continuity is violated?

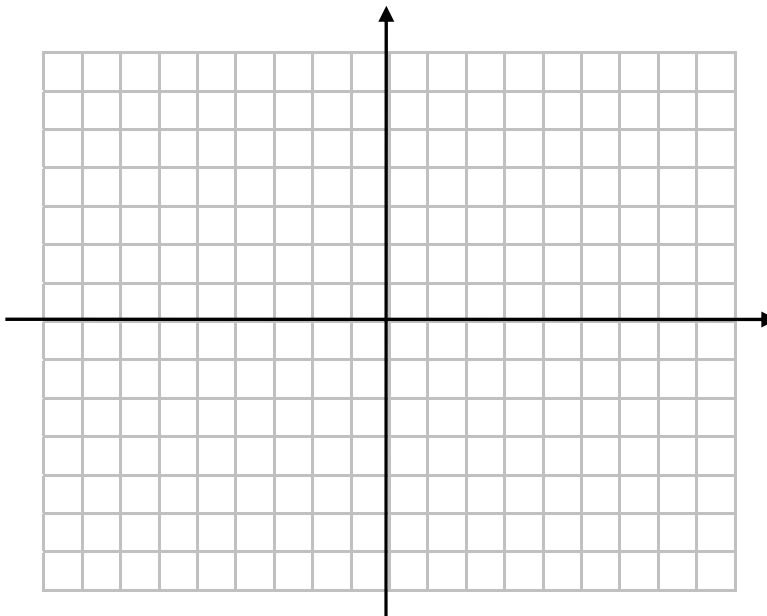


Continuity at an endpoint of the domain:

A function f is continuous at a number a , a left endpoint of its domain, if $\lim_{x \rightarrow a^+} f(x) = f(a)$.

A function f is continuous at a number a , a right endpoint of its domain, if $\lim_{x \rightarrow a^-} f(x) = f(a)$.

Example 9: $f(x) = \sqrt{4-x}$

**One-sided continuity:**

Definition: A function f is continuous from the left at a number a if

$$\lim_{x \rightarrow a^-} f(x) = f(a)$$

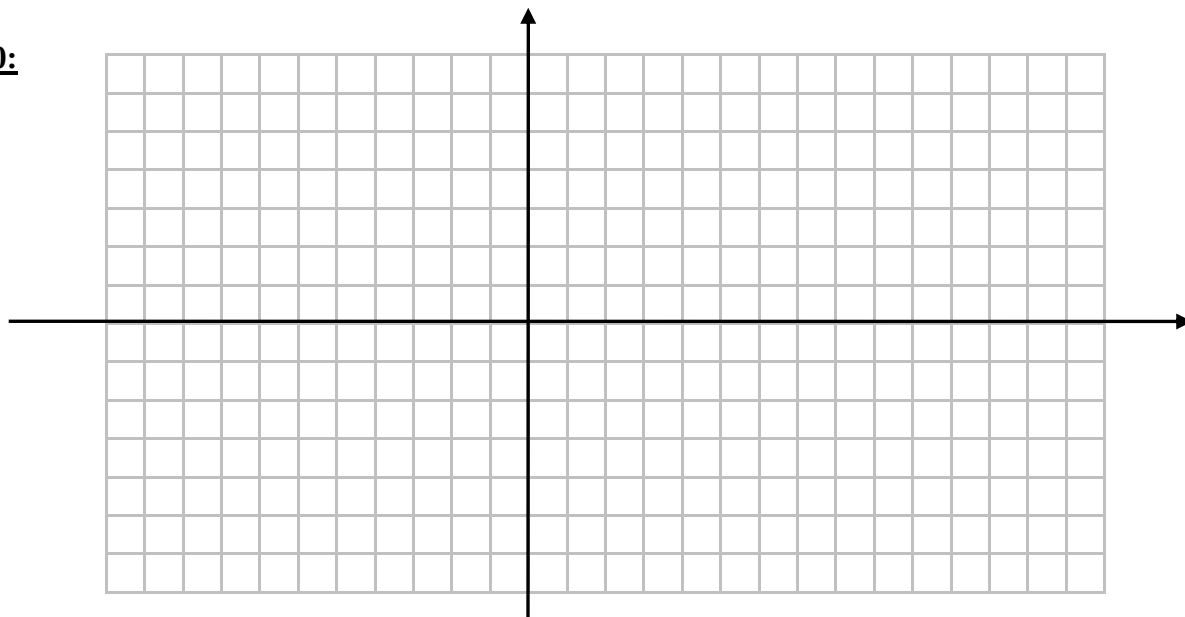
A function f is continuous from the right at a number a if

$$\lim_{x \rightarrow a^+} f(x) = f(a)$$

Continuity on an interval:

Definition: A function f is continuous on an interval if it is continuous at every point in the interval.

(If f is defined only at one side of an endpoint, then only continuity from the left or right is needed for it to be continuous at the endpoint.)

Example 10:

On what intervals is the above function continuous?

Theorem: If f and g are continuous at a and c is a constant, then $f + g$, $f - g$, fg , cf are also continuous at a . The quotient $\frac{f}{g}$ is also continuous at a if $g(a) \neq 0$.

Theorem:

Polynomials, rational functions, root functions, and trigonometric functions are continuous at every number in their domains.

Theorem:

If f is continuous at $b = \lim_{x \rightarrow a} g(x)$, then $\lim_{x \rightarrow a} f(g(x)) = f(\lim_{x \rightarrow a} g(x))$.

Theorem:

If g is continuous at a and f is continuous at $g(a)$, then the composite function $f \circ g$ given by $(f \circ g)(x) = f(g(x))$ is continuous at a .

Example 11: Where is $f(x) = \sin\left(\frac{1}{x}\right)$ continuous? At each discontinuity, classify the type of discontinuity and state the condition for continuity that is violated.

Example 12: Where is $f(x) = \sqrt{\frac{x^2+1}{(x-1)^3}}$ continuous? At each discontinuity, classify the type of discontinuity and state the condition for continuity that is violated.

Example 13: Determine the values of x , if any, at which the function is discontinuous. For each, classify the type of discontinuity and state the condition for continuity that is violated.

$$f(x) = \frac{5x^3 - 8x^2}{x - 7}$$

Example 14: Determine the values of x , if any, at which the function is discontinuous. For each, classify the type of discontinuity and state the condition for continuity that is violated.

$$f(x) = \frac{x^3 - 3x^2}{x^2 + 4}$$

Example 15: Determine the values of x , if any, at which the function is discontinuous. For each, classify the type of discontinuity and state the condition for continuity that is violated. At which of these numbers is it continuous from the left or right?

$$f(x) = \begin{cases} x+3 & \text{if } x > 0 \\ 4 & \text{if } x = 0 \\ x^2 + 3 & \text{if } x < 0 \end{cases}$$

Example 16: Determine the values of x , if any, at which the function is discontinuous. For each, classify the type of discontinuity and state the condition for continuity that is violated. At which of these numbers is it continuous from the left or right?

$$f(x) = \begin{cases} 5x & \text{if } x > 1 \\ 5 & \text{if } x = 1 \\ x+5 & \text{if } x < 1 \end{cases}$$

Example 17: Determine the values of x , if any, at which the function is discontinuous.

$$g(x) = \begin{cases} 1 & x \text{ rational} \\ 0 & x \text{ irrational} \end{cases}$$

Example 18: Determine the values of x , if any, at which the function is discontinuous. For each, classify the type of discontinuity and state the condition for continuity that is violated.

$$f(x) = \begin{cases} \frac{|x|}{x} & x \neq 0 \\ 3 & x = 0 \end{cases}$$

Example 19: Determine the values of x , if any, at which the function is discontinuous. For each, classify the type of discontinuity and state the condition for continuity that is violated.

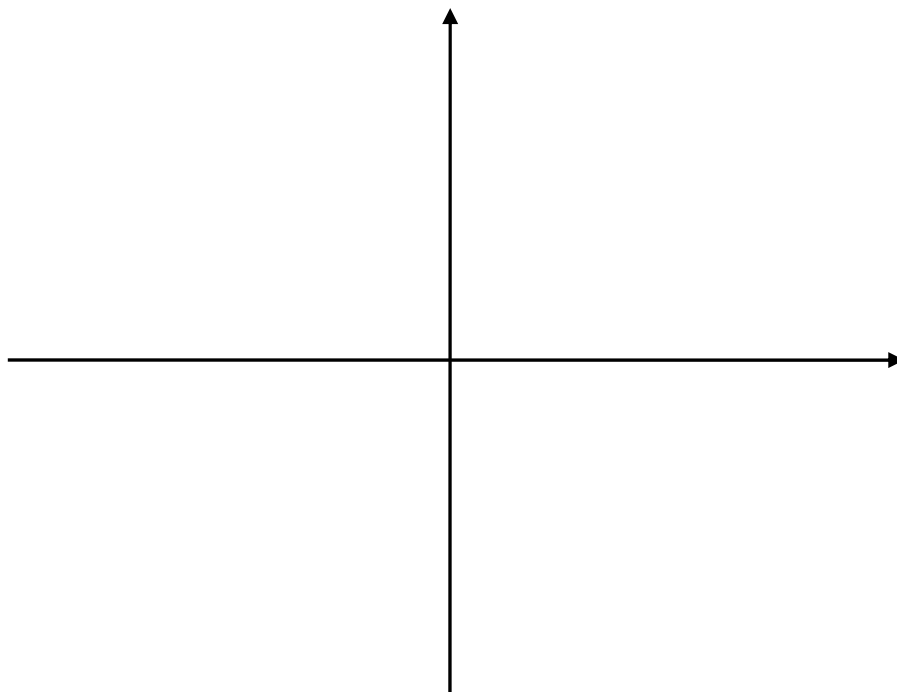
$$f(x) = \begin{cases} \frac{x^2 - 25}{x - 5} & x \neq 5 \\ 10 & x = 5 \end{cases}$$

Example 20: Find a function g that agrees with f for $x \neq 1$ and is continuous on $(-\infty, \infty)$.

$$f(x) = \frac{x^2 - 1}{x - 1}.$$

The Intermediate Value Theorem:

Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in (a, b) such that $f(c) = N$.



Example 21: Show that $f(x) = x^3 - 4x^2 + 6$ has a zero between 1 and 2.

Example 22: Is there a number that is equal to its own cosine?