1.4: Continuity and One-Sided Limits

One-Sided Limits:

$\lim_{x\to a^-} f(x) = L$ means that $f(x)$ approaches L as x approaches a from the left.

$\lim_{x\to a^-} f(x) = L$

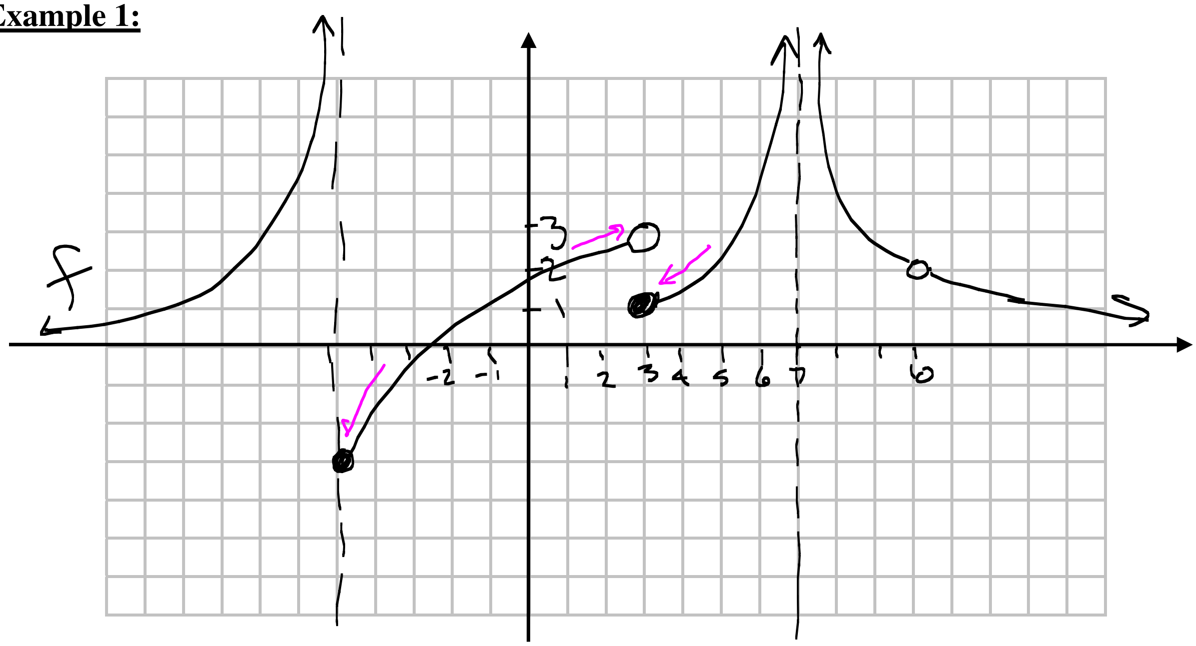
$\lim_{x\to a^+} f(x) = L$ means that $f(x)$ approaches L as x approaches a from the right.

$\lim_{x\to a^+} f(x) = L$

$\lim_{x\to a} f(x) = L$ if and only if $\lim_{x\to a^-} f(x) = L$ and $\lim_{x\to a^+} f(x) = L$.

$\lim_{x\to a} f(x) = L$ iff $\lim_{x\to a^-} f(x) = L$ and $\lim_{x\to a^+} f(x) = L$

Example 1:



$\lim_{x\to 3^-} f(x) = \underline{\quad 3 \quad}$

$\lim_{x\to 3^+} f(x) = \underline{\quad 1 \quad}$

So $\lim_{x\to 3} f(x)$ does not exist.

$\left.\begin{array}{l}\lim_{x\to 7^-} f(x) = \infty \\ \lim_{x\to 7^+} f(x) = \infty\end{array}\right\}$ so $\lim_{x\to 7} f(x) = \infty$

none of these 3 limits exist!

$\lim_{x\to -5^-} f(x) = \infty$ (so this limit does not exist)

$\lim_{x\to -5^+} f(x) = -3$

$\lim_{x\to -5} f(x)$ does not exist.

$\lim_{x\to 10^-} f(x) = \lim_{x\to 10^+} f(x) = \lim_{x\to 10} f(x)$
$= 2$

Example 2: Determine $\lim_{x \to 1^-} f(x)$, $\lim_{x \to 1^+} f(x)$, and $\lim_{x \to 1} f(x)$, and $\lim_{x \to 2} f(x)$.

$$f(x) = \begin{cases} x^2 & \text{if } x \le 1 \\ x-3 & \text{if } x > 1 \end{cases}.$$

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} (x-3) = 2-3 = \boxed{-1}$$

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x^2) = 1^2 = \boxed{1}$$

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (x-3) = 1-3 = \boxed{-2}$$

$\lim_{x \to 1} f(x)$ does not exist, because $\lim_{x \to 1^-} f(x) \neq \lim_{x \to 1^+} f(x)$.

Example 3: Determine $\lim_{x \to -2^-} f(x)$, $\lim_{x \to -2^+} f(x)$, and $\lim_{x \to -2} f(x)$.

$$f(x) = \begin{cases} 1-x^3 & \text{if } x \le -2 \\ 7-x & \text{if } x > -2 \end{cases}.$$

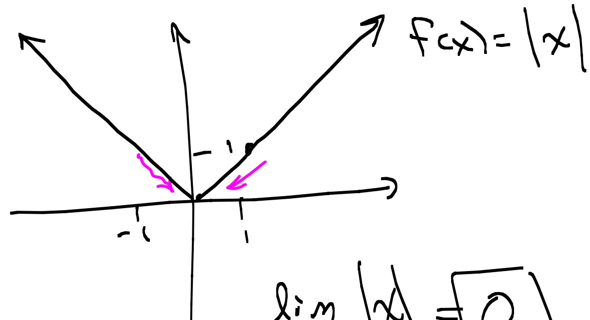
$$\lim_{x \to -2^-} f(x) = 1-(-2)^3 = 1-(-8) = \boxed{9}$$

$$\lim_{x \to -2^+} f(x) = 7-(-2) = \boxed{9}$$

$$\lim_{x \to -2} f(x) = \boxed{9}$$

Example 4: Determine $\lim_{x \to 0} |x|$, if it exists.

From graph:



$$\lim_{x \to 0} |x| = \boxed{0}$$

OR, write as a piecewise function:

$$f(x) = |x| = \begin{cases} x & \text{if } x \ge 0 \\ -x & \text{if } x < 0 \end{cases}$$

$$\left.\begin{array}{l} \lim_{x \to 0^-} |x| = \lim_{x \to 0^-} (-x) = -0 = 0 \\ \lim_{x \to 0^+} |x| = \lim_{x \to 0^+} (x) = 0 \end{array}\right\} \text{so } \lim_{x \to 0} |x| = \boxed{0}$$

Example 5: Determine $\lim\limits_{x\to 1} f(x)$ and $\lim\limits_{x\to -2} f(x)$, where $f(x)=\begin{cases}\sqrt{5-x} & \text{if } x\geq 1\\ 3x^2-7 & \text{if } x<1\end{cases}$.

$$\left.\begin{aligned}\lim_{x\to 1^-} f(x) &= 3(1)^2-7 = -4\\ \lim_{x\to 1^+} f(x) &= \sqrt{5-1} = \sqrt{4} = 2\end{aligned}\right\}\text{ not equal, so } \lim_{x\to 1} f(x) \text{ does not exist.}$$

$$\lim_{x\to -2} f(x) = 3(-2)^2-7 = 12-7 = \boxed{5}$$

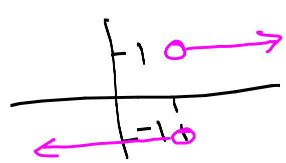
Example 6: Determine $\lim\limits_{x\to 0} f(x)$, where $f(x)=\begin{cases}\cos x & \text{if } x<0\\ 2 & \text{if } x=0\\ \sqrt{x+1} & \text{if } x>0\end{cases}$.

$$\left.\begin{aligned}\lim_{x\to 0^-} f(x) &= \cos 0 = 1\\ \lim_{x\to 0^+} f(x) &= \sqrt{0+1} = 1\end{aligned}\right\}\text{ so } \lim_{x\to 0} f(x) = \boxed{1}$$

Example 7: Determine $\lim\limits_{x\to 1}\dfrac{|x-1|}{x-1}$.

$$\frac{|x-1|}{x-1} = \begin{cases}\dfrac{x-1}{x-1} = 1 & \text{if } x-1>0 \quad (\text{so } x>1)\\[2ex] \dfrac{-(x-1)}{x-1} = -1 & \text{if } x-1<0 \quad (\text{so } x<1)\end{cases}$$

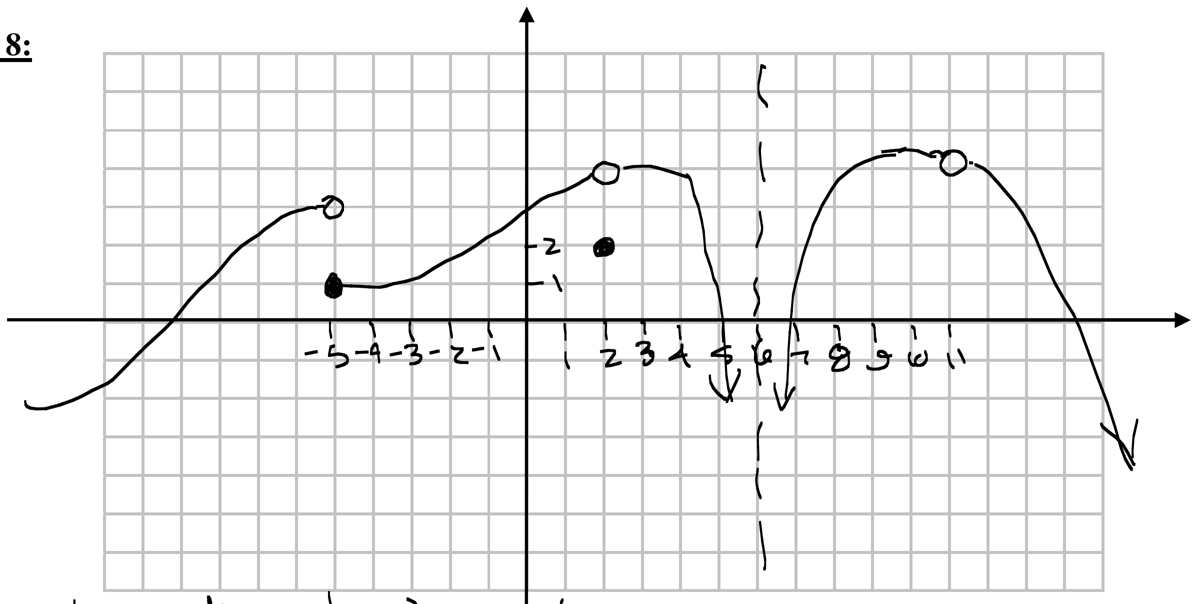
$$\boxed{\lim_{x\to 1}\frac{|x-1|}{x-1} \text{ does not exist.}}$$



Continuity of a function:

In most cases, we can think of a continuous function as one that can be drawn "without lifting your pencil from the paper". In other words, there are no holes, breaks, or jumps.

Example 8:



This function has discontinuities at $-5, 2, 6, 11$

Continuous on each of these intervals: $(-\infty,-5), (-5,2), (2,6), (6,11), (11,\infty)$

Definition: A function f is continuous at a number a, an interior point of its domain, if

$$\lim_{x\to a} f(x) = f(a)$$

Conditions for Continuity:

In order for f to be continuous at a, all the following conditions must hold:

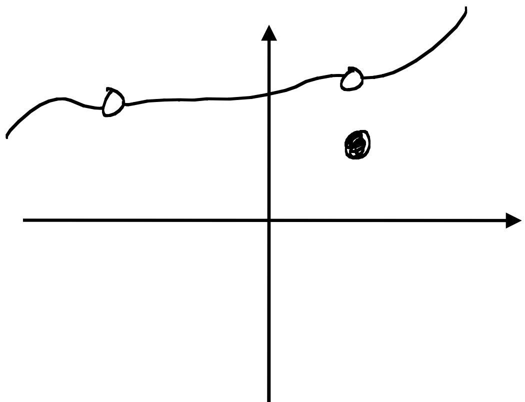
1. $f(a)$ is defined.
2. $\lim_{x\to a} f(x)$ exists.
3. $\lim_{x\to a} f(x) = f(a)$.

<u>Types of discontinuities</u>:

1. Removable discontinuity
2. Infinite discontinuity
3. Jump discontinuity
4. Oscillating discontinuity

$\left.\begin{array}{l}2.\\3.\\4.\end{array}\right\}$ nonremovable

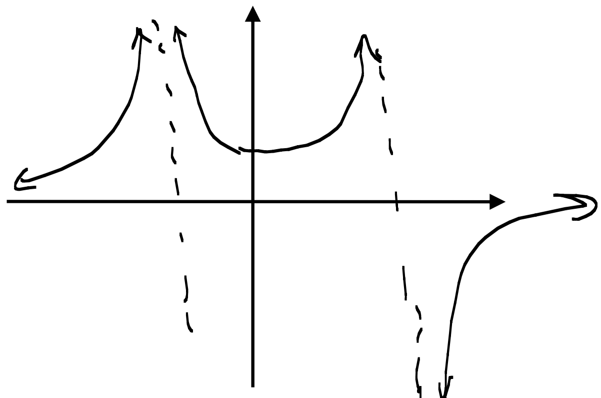
What do these look like? For each type of discontinuity, what condition of continuity is violated?



Removable Discontinuities

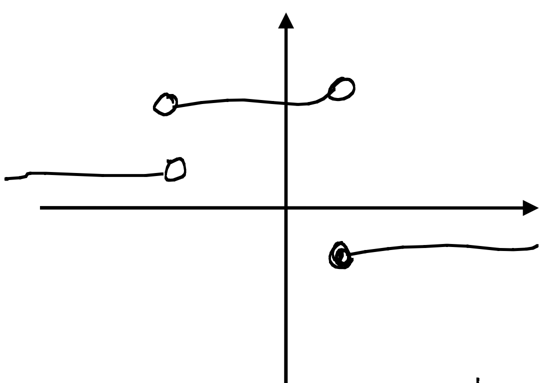
$\lim_{x\to a} f(x)$ exists, and $f(a)$ may exist

but $\lim_{x\to a} f(x) \neq f(a)$



Infinite Discontinuities

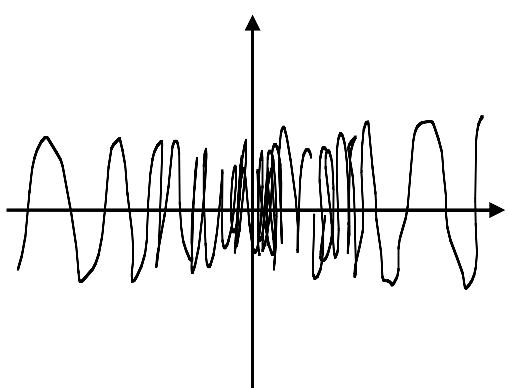
$\lim_{x\to a} f(x)$ does not exist



Jump discontinuities.

$\lim_{x\to a} f(x)$ does not exist because

$\lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} f(x)$



ex: $y = \sin\left(\frac{1}{x}\right)$

Oscillating discontinuities

($\lim_{x\to a} f(x)$ does not exist)

Continuity at an endpoint of the domain:

A function f is <u>continuous at a number a</u>, a left endpoint of its domain, if $\lim_{x\to a^+} f(x) = f(a)$.

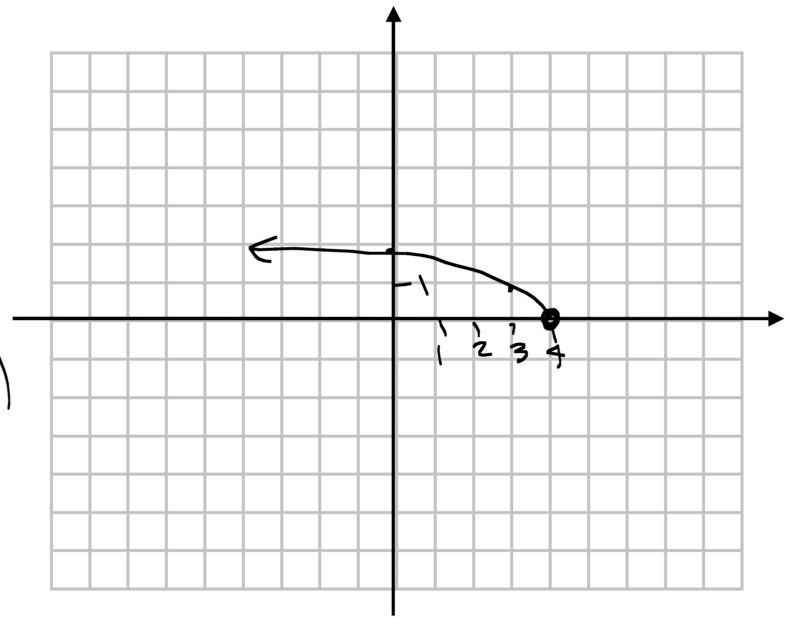
A function f is <u>continuous at a number a</u>, a right endpoint of its domain, if $\lim_{x\to a^-} f(x) = f(a)$.

<u>Example 9:</u> $f(x) = \sqrt{4-x}$

Domain: $(-\infty, 4]$

This function is continuous at 4. (because $\lim_{x\to 4^-} \sqrt{4-x} = f(4)$)

So f is continuous on $(-\infty, 4]$.



One-sided continuity:

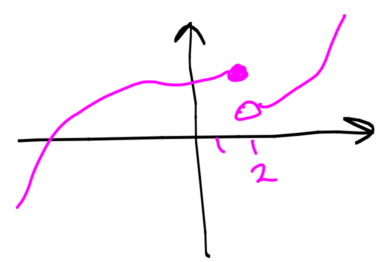
<u>Definition</u>: A function f is continuous from the left at a number a if

$$\lim_{x\to a^-} f(x) = f(a)$$

A function f is continuous from the right at a number a if

$$\lim_{x\to a^+} f(x) = f(a)$$

<u>Ex:</u>



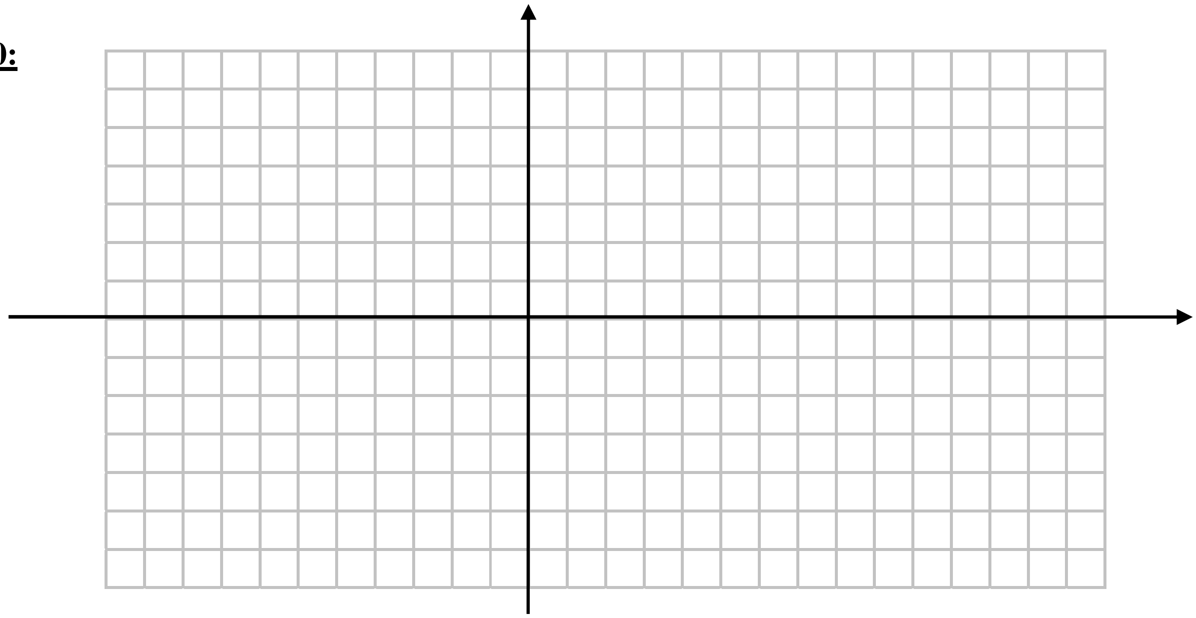
This function is continuous from the left at 2, because $\lim_{x\to 2^-} f(x) = f(2)$

Continuity on an interval:

<u>Definition</u>: A function f is continuous on an interval if it is continuous at every point in the interval.

(If f is defined only at one side of an endpoint, then only continuity from the left or right is needed for it to be continuous at the endpoint.)

Example 10:



On what intervals is the above function continuous?

Theorem: If f and g are continuous at a and c is a constant, then $f+g$, $f-g$, fg, cf are also continuous at a. The quotient $\frac{f}{g}$ is also continuous at a if $g(a) \neq 0$.

Theorem:
Polynomials, rational functions, root functions, and trigonometric functions are continuous at every number in their domains.

Theorem:
If f is continuous at $b = \lim_{x\to a} g(x)$, then $\lim_{x\to a} f(g(x)) = f(\lim_{x\to a} g(x))$.

Theorem:
If g is continuous at a and f is continuous at $g(a)$, then the composite function $f \circ g$ given by $(f \circ g)(x) = f(g(x))$ is continuous at a.

Example 11: Where is $f(x) = \sin\left(\frac{1}{x}\right)$ continuous? At each discontinuity, classify the type of discontinuity and state the condition for continuity that is violated.

Continuous except at 0.
Oscillating discontinuity (limit doesn't exist)

Example 12: Where is $f(x) = \sqrt{\frac{x^2+1}{(x-1)^3}}$ continuous? At each discontinuity, classify the type of discontinuity and state the condition for continuity that is violated.

Domain: numerator $x^2+1 \geq 0$ for all x.
For this function to be defined, the denominator must be positive: $x-1>0$
$x>1$

Domain: $(1, \infty)$

This function is continuous on its domain

Example 13: Determine the values of x, if any, at which the function is discontinuous. For each, classify the type of discontinuity and state the condition for continuity that is violated.

$$f(x) = \frac{5x^3 - 8x^2}{x-7}$$

Discontinuous at 7
$f(7)$ does not exist.

Domain: $(-\infty, 7) \cup (7, \infty)$

infinite discontinuity.

Also, $\lim_{x\to 7} f(x)$ does not exist.

Example 14: Determine the values of x, if any, at which the function is discontinuous. For each, classify the type of discontinuity and state the condition for continuity that is violated.

Denominator $x^2 + 4 > 0$ for all x.

$$f(x) = \frac{x^3 - 3x^2}{x^2 + 4}$$

$x^2 + 4 = 0$

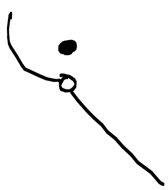
$x^2 = -4$

$x = \pm 2i$

Domain: $(-\infty, \infty)$

$\boxed{\text{Continuous everywhere}}$.

Example 15: Determine the values of x, if any, at which the function is discontinuous. For each, classify the type of discontinuity and state the condition for continuity that is violated. At which of these numbers is it continuous from the left or right?



$$f(x) = \begin{cases} x+3 & \text{if } x > 0 \\ 4 & \text{if } x = 0 \\ x^2+3 & \text{if } x < 0 \end{cases}.$$

$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (x^2 + 3)$

$= 0^2 + 3 = 3$

$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (x+3)$

$= 0 + 3 = 3$

So $\lim_{x \to 0} f(x) = 3$.

$f(0) = 4$

Because $f(0) \neq \lim_{x \to 0} f(x)$, there is a $\boxed{\text{discontinuity at } 0}$.

This is a $\boxed{\text{removable discontinuity}}$.

Example 16: Determine the values of x, if any, at which the function is discontinuous. For each, classify the type of discontinuity and state the condition for continuity that is violated. At which of these numbers is it continuous from the left or right?

$$f(x) = \begin{cases} 5x & \text{if } x > 1 \\ 5 & \text{if } x = 1 \\ x+5 & \text{if } x < 1 \end{cases}.$$

$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x + 5) = 1 + 5 = 6$

$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (5x) = 5(1) = 5$

Because $\lim_{x \to 1^-} f(x) \neq \lim_{x \to 1^+} f(x)$, the limit $\lim_{x \to 1} f(x)$ does not exist

$\boxed{\text{Discontinuity at } x=1}$

$\boxed{\text{Jump discontuity}}$

Example 17: Determine the values of x, if any, at which the function is discontinuous.

$$g(x) = \begin{cases} 1 & x \text{ rational} \\ 0 & x \text{ irrational} \end{cases}$$

Discontinuous everywhere

Domain: $(-\infty, \infty)$

Example 18: Determine the values of x, if any, at which the function is discontinuous. For each, classify the type of discontinuity and state the condition for continuity that is violated.

$$f(x) = \begin{cases} \dfrac{|x|}{x} & x \neq 0 \\ 3 & x = 0 \end{cases}$$

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \left(\frac{|x|}{x} \right) = \lim_{x \to 0^-} \frac{-x}{x} = -1$$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \left(\frac{|x|}{x} \right) = \lim_{x \to 0^+} \frac{x}{x} = 1$$

Jump discontinuity at 0.

Discontinuous because $\lim_{x \to 0} f(x)$ does not exist.

Example 19: Determine the values of x, if any, at which the function is discontinuous. For each, classify the type of discontinuity and state the condition for continuity that is violated.

$$f(x) = \begin{cases} \dfrac{x^2 - 25}{x - 5} & x \neq 5 \\ 10 & x = 0 \end{cases}$$

Example 20: Find a function g that agrees with f for $x \neq 1$ and is continuous on $(-\infty, \infty)$.

$$f(x) = \frac{x^2 - 1}{x - 1}.$$

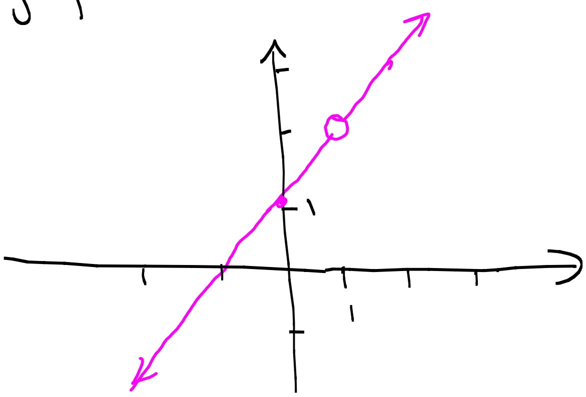
$f(x) = \frac{(x+1)(x-1)}{x-1}$ Domain of f: $x \neq 1$

$(-\infty, 1) \cup (1, \infty)$

$\boxed{g(x) = x + 1}$ (obtained by "cancelling" $x-1$ in f)

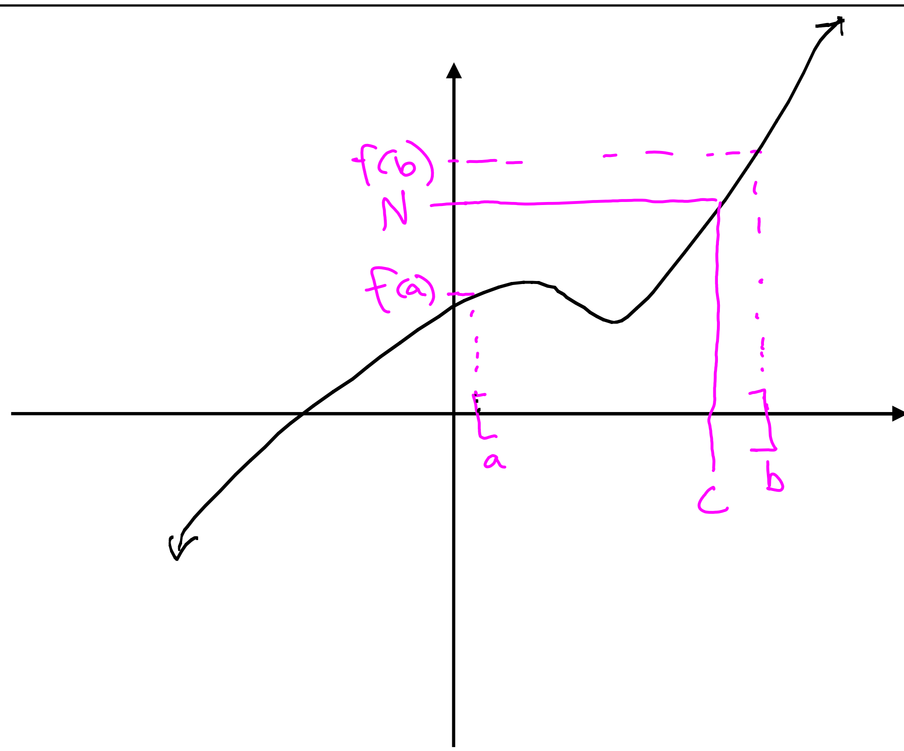
Domain of g: $(-\infty, \infty)$

Let's graph f:



The Intermediate Value Theorem:

Suppose that *f* is continuous on the closed interval $[a,b]$ and let *N* be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number *c* in (a,b) such that $f(c) = N$.



Example 21: Show that $f(x) = x^3 - 4x^2 + 6$ has a zero between 1 and 2.

$$f(1) = 1^3 - 4(1)^2 + 6 = 1 - 4 + 6 = 3$$

$$f(2) = 2^3 - 4(2)^2 + 6 = 8 - 16 + 6 = 14 - 16 = -2$$

f is continuous on $(-\infty, \infty)$ and so certainly continuous on $[-2, 3]$.

0 is between $f(1) = 3$ and $f(2) = -2$.

So, from IVT (Int. Value Thm), there must be a c in $[-2, 3]$ such that $f(c) = 0$.

Example 22: Is there a number that is equal to its own cosine?

in other words, does $x = \cos x$ have a solution?

Let $f(x) = \cos x - x$. Does $f(x)$ have a zero?

Is f continuous? Yes, continuous on $(-\infty, \infty)$

$$f(\pi) = \cos\pi - \pi = -1 - \pi \approx -1 - 3.14 \approx -4.14$$

$$f(2\pi) = \cos 2\pi - 2\pi = 1 - 2\pi \approx 1 - 6.28 \approx -5.28$$

$$f(0) = \cos 0 - 0 = 1 - 0 = 1$$

0 is between -4.14 and 1, so there must be a c in $[0, \pi]$ such that $f(x) = 0$.

So $\boxed{\text{Yes!}}$

if $f(x) = 0 = \cos x - x$ then $x = \cos x$.