2.1: The Derivative and the Tangent Line Problem

What is the definition of a "tangent line to a curve"?

To answer the difficulty in writing a clear definition of a tangent line, we can define it as the limiting position of the secant line as the second point approaches the first.

Definition: The tangent line to the curve $y = f(x)$ at the point $(a, f(a))$ is the line through P with slope

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$ provided this limit exists.

Equivalently,

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$ provided this limit exists.

Note: If the tangent line is vertical, this limit does not exist. In the case of a vertical tangent, the equation of the tangent line is $x = a$.

Note: The slope of the tangent line to the graph of f at the point $(a, f(a))$ is also called the slope of the graph of f at $x = a$.

How to get the second expression for slope: Instead of using the points $(a, f(a))$ and $(x, f(x))$ on the secant line and letting $x \to a$, we can use $(a, f(a))$ and $(a+h, f(a+h))$ and let $h \to 0$.

Example 1: Find the slope of the curve $y = 4x^2 + 1$ at the point $(3, 37)$. Find the equation of the tangent line at this point.

Example 2: Find an equation of the tangent line to the curve $y = x^3$ at the point $(1,1)$.

Example 3: Determine the equation of the tangent line to $f(x) = \sqrt{x}$ at the point where $x = 2$.

The derivative:

The derivative of a function at x is the slope of the tangent line at the point $(x, f(x))$. It is also the instantaneous rate of change of the function at x.

<u>Definition</u>: The *derivative* of a function f at x is the function $f\,'$ whose value at x is given by

$$f\,'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$, provided this limit exists.

The process of finding derivatives is called <u>differentiation</u>. To <u>differentiate</u> a function means to find its derivative.

Equivalent ways of defining the derivative:

$$f\,'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$ (Our book uses this one. It is identical to the definition above, except uses Δx in place of h.)

$$f\,'(x) = \lim_{w \to x} \frac{f(w) - f(x)}{w - x}$$

$$f\,'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$ (Gives the derivative at the specific point where $x = a$.)

$$f\,'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$ (Gives the derivative at the specific point where $x = a$.)

<u>Example 4:</u> Suppose that $g(x) = \dfrac{x^2 - 6x}{3}$. Determine $g\,'(x)$ and $g\,'(3)$.

Example 5: Suppose that $f(x) = \sqrt{x^2 + 1}$. Find the equation of the tangent line at the point where $x = 2$.

Example 6: Determine the equation of the tangent line to $f(x) = \dfrac{x-2}{x^2+1}$ at the point $\left(-2, -\dfrac{4}{5}\right)$.

Summary:

The slope of the secant line between two points is often called a difference quotient. The difference quotient of *f* at *a* can be written in either of the forms below.

$$\frac{f(x)-f(a)}{x-a} \qquad\qquad \frac{f(a+h)-f(a)}{h}.$$

Both of these give the slope of the secant line between two points: $(x, f(x))$ and $(a, f(a))$ or, alternatively, $(a, f(a))$ and $(a+h, f(a+h))$.
The slope of the secant line is also the average rate of change of *f* between the two points.

The derivative of *f* at *a* is:

1) the limit of the slopes of the secant lines as the second point approaches the point $(a, f(a))$.

2) the slope of the tangent line to the curve $y = f(x)$ at the point where $x = a$.

3) the (instantaneous) rate of change of f with respect to *x* at *a*.

4) $\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$ (limit of the difference quotient)

5) $\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$ (limit of the difference quotient)

Common notations for the derivative of $y = f(x)$:

$f'(x) \qquad \frac{d}{dx}f(x) \qquad y' \qquad D_x f(x) \qquad \frac{dy}{dx} \qquad Df(x)$

The notation $\frac{dy}{dx}$ was created by Gottfried Wilhelm Leibniz and means $\frac{dy}{dx} = \lim_{\Delta x\to 0}\frac{\Delta x}{\Delta y}$.
To evaluate the derivative at a particular number *a*, we write

$$f'(a) \text{ or } \left.\frac{dy}{dx}\right|_{x=a}$$

Differentiability:

Definition: A function *f* is *differentiable* at *a* if $f'(a)$ exists. It is *differentiable on an open interval* if it is differentiable at every number in the interval.

Theorem: If *f* is differentiable at *a*, then *f* is continuous at *a*.

Note: The converse is not true—there are functions that are continuous at a number but not differentiable.

Note: Open intervals: (a,b), $(-\infty,a)$, (a,∞), $(-\infty,\infty)$.

Closed intervals: $[a,b]$, $(-\infty,a]$, $[a,\infty)$, $(-\infty,\infty)$.

To discuss differentiability on a closed interval, we need the concept of a *one-sided derivative*.

Derivative from the left: $\lim_{x\to a^-}\frac{f(x)-f(a)}{x-a}$

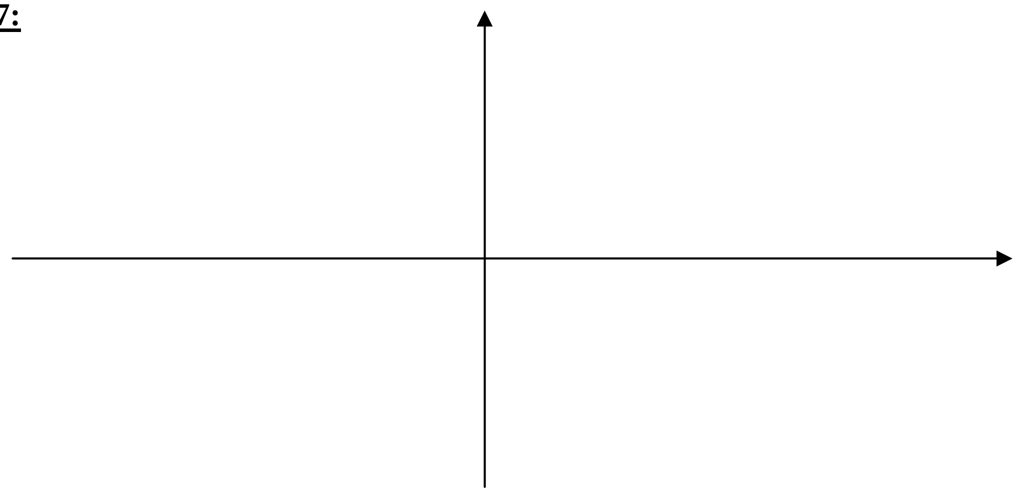
Derivative from the right: $\lim_{x\to a^+}\frac{f(x)-f(a)}{x-a}$

For a function *f* to be differentiable on the closed interval $[a,b]$, it must be differentiable on the open interval (a,b). In addition, the derivative from the right at *a* must exist, and the derivative from the left at *b* must exist.

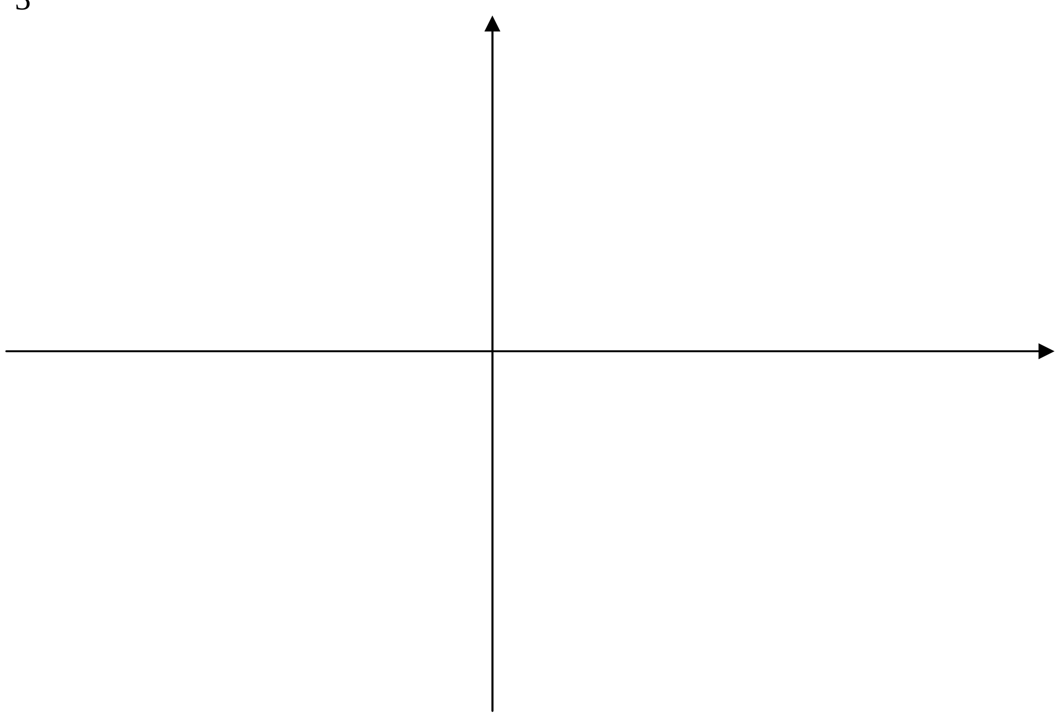
Ways in which a function can fail to be differentiable:

1. Sharp corner
2. Cusp
3. Vertical tangent
4. Discontinuity

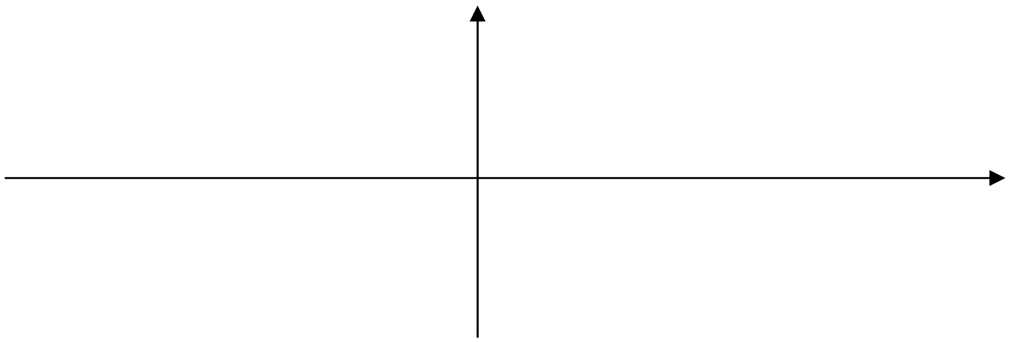
Example 7:

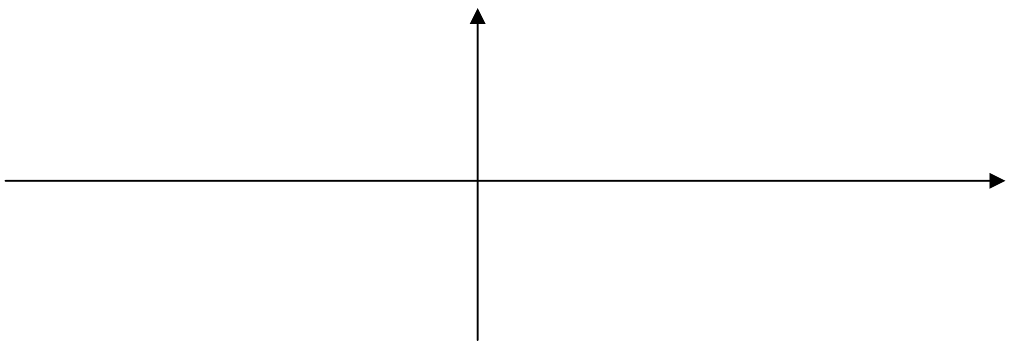


Example 8: Sketch the graph of a function for which $f(0)=2$, $f'(0)=-1$, $f(2)=1$, $f'(2)=\frac{1}{3}$, $f'(3)>f'(2)$, and $f'(5)<0$.



Example 9: Use the graph of the function to draw the graph of the derivative.





Example 10: Use the graph of the function to draw the graph of the derivative.