

To answer the difficulty in writing a clear definition of a tangent line, we can define it as the limiting position of the secant line as the second point approaches the first.

Definition: The tangent line to the curve $y=f(x)$ at the point $(a, f(a))$ is the line through $P$ with slope

$$
m=\lim _{x \rightarrow a} \frac{f(x)-f(a)}{x-a} \text { provided this limit exists. }
$$

Equivalently,

$$
m=\lim _{h \rightarrow 0} \frac{f(a+h)-f(a)}{h} \text { provided this limit exists. }
$$

Note: If the tangent line is vertical, this limit does not exist. In the case of a vertical tangent, the equation of the tangent line is $x=a$.

Note: The slope of the tangent line to the graph of $f$ at the point $(a, f(a))$ is also called the slope of the graph of $f$ at $x=a$.

How to get the second expression for slope: Instead of using the points ( $a, f(a)$ ) and ( $x, f(x)$ ) on the secant line and letting $x \rightarrow a$, we can use $(a, f(a))$ and $(a+h, f(a+h))$ and let $h \rightarrow 0$.

Find egg of tangent line: slope:m $=24$, $\operatorname{Point}(x, y)=,(3,37)$

$$
\begin{aligned}
& y-y_{1}=m\left(x-x_{1}\right) \\
& y-37=24(x-3) \\
& y-37=24 x-72
\end{aligned} \quad \begin{aligned}
& y=24 x-72+37 \\
& y=24 x-35
\end{aligned}
$$

Example 1: Find the slope of the curve $y=4 x^{2}+1$ at the point $(3,37)$. Find the equation of

$$
\begin{aligned}
m= & \lim _{h \rightarrow 0} \frac{f(3+h)-f(3)}{h}=\lim _{h \rightarrow 0} \frac{4(3+h)^{2}+1-\left[4(3)^{2}+1\right]}{h} \\
& =\lim _{h \rightarrow 0} \frac{4\left(9+6 h+h^{2}\right)+1-37}{h}=\lim _{h \rightarrow 0} \frac{36+24 h+4 h^{2}-3 \%}{h} \\
& =\lim _{h \rightarrow 0} \frac{24 h+4 h^{2}}{h}=\lim _{h \rightarrow 0} \frac{h(24+4 h)}{h} \\
& =\lim _{h \rightarrow 0}(24+4 h)=24+4(0)=24 \text { of curve }
\end{aligned}
$$

$$
\begin{aligned}
& \text { Example 2: Find an equation of the tangent line to the curve } y=x^{3} \text { at the point }(1,1) \text {. } \\
& \text {-Alternative definition: } \\
& \lim _{x \rightarrow a} \frac{f(x)-f(a)}{x-a} \lim _{x \rightarrow 1} \frac{f(x)-f(1)}{x-1}=\lim _{x \rightarrow 1} \frac{x^{3}-1}{x-1} \\
& =\lim _{x \rightarrow 1} \frac{(x-1)\left(x^{2}+\mid x+1\right)}{x-x}=\lim _{x \rightarrow 1}\left(x^{2}+x+1\right)=1^{2}+1+1 \\
& =3=\text { slope } \\
& \text { Find en: } \\
& y-y_{1}=m\left(x-x_{1}\right) \\
& \text { here: } a=1 \\
& y-1=3(x-1) \\
& y=3 \overline{x-2} \\
& \text { Example 3: Determine the equation of the tangent line to } f(x)=\sqrt{x} \text { at the point where } x=2 \text {. }
\end{aligned}
$$

$$
\begin{aligned}
& m=\lim _{h \rightarrow 0} \frac{f(2+h)-f(2)}{h}=\lim _{h \rightarrow 0} \frac{\sqrt{2+h}-\sqrt{2}}{h}=\lim _{h \rightarrow 0}\left[\frac{\sqrt{2+h}-\sqrt{2}}{h} \cdot \frac{\sqrt{2+h}+\sqrt{2}}{\sqrt{2+h}+\sqrt{2}}\right]\left(\frac{f(2)}{}=\sqrt{2}\right. \\
&= \lim _{h \rightarrow 0} \frac{2+h-2}{h(\sqrt{2+h}+\sqrt{2})}=\lim _{h \rightarrow 0} \frac{h}{w(\sqrt{2+h}+\sqrt{2})}(a-b)(a+b) \\
&= \lim _{h \rightarrow 0} \frac{1}{\sqrt{2+h}+\sqrt{2}}=\frac{1}{\sqrt{2+0}+\sqrt{2}}=\frac{1}{2 \sqrt{2}}=\frac{1}{2 \sqrt{2}}\left(\frac{\sqrt{2}}{\sqrt{2}}\right) \\
&=\frac{\sqrt{2}}{4} \\
& \text { Find eqn: } \quad \begin{aligned}
y-y_{1} & =m\left(x-x_{1}\right) \\
y-\sqrt{2} & =\frac{\sqrt{2}}{4}(x-2) \\
y & =\frac{\sqrt{2}}{4} x-\frac{2 \sqrt{2}}{4}+\sqrt{2} \\
y & =\frac{\sqrt{2}}{4} x-\frac{\sqrt{2}}{2}+\frac{2 \sqrt{2}}{2}
\end{aligned}
\end{aligned}
$$

The derivative:
The derivative of a function at $x$ is the slope of the tangent line at the point $(x, f(x))$. It is also the instantaneous rate of change of the function at $x$.

Definition: The derivative of a function $f$ at $x$ is the function $f^{\prime}$ whose value at $x$ is given by

$$
f^{\prime}(x)=\lim _{h \rightarrow 0} \frac{f(x+h)-f(x)}{h}, \text { provided this limit exists. }
$$

The process of finding derivatives is called differentiation. To differentiate a function means to find its derivative.

Equivalent ways of defining the derivative:

$$
\begin{aligned}
& f^{\prime}(x)=\lim _{\Delta x \rightarrow 0} \frac{f(x+\Delta x)-f(x)}{\Delta x} \\
& f^{\prime}(x)=\lim _{w \rightarrow x} \frac{f(w)-f(x)}{w-x} \\
& f^{\prime}(a)=\lim _{x \rightarrow a} \frac{f(x)-f(a)}{x-a} \\
& f^{\prime}(a)=\lim _{h \rightarrow 0} \frac{f(a+h)-f(a)}{h}
\end{aligned}
$$

(Our book uses this one. It is identical to the

$$
\text { definition above, except uses } \Delta x \text { in place of } h .)
$$

(Gives the derivative at the specific point where $x=a$.)
(Gives the derivative at the specific point where $x=a$.)

Example 4: Suppose that $g(x)=\frac{x^{2}-6 x}{3}$. Determine $g^{\prime}(x)$ and $g^{\prime}(3)$.

$$
\begin{aligned}
& \text { '(3). } \begin{aligned}
& g(x)=\frac{x^{2}}{3}-\frac{6 x}{3} \\
&=\frac{1}{3} x^{2}-2 x \\
&(x+h)-\left(\frac{1}{3} x^{2}-2 x\right)
\end{aligned} h
\end{aligned}
$$


$=\lim _{h \rightarrow 0} \frac{\frac{1}{3} x^{2}+\frac{2}{3} x h+\frac{1}{3} h^{2}-2 h-\frac{1}{3} x^{2}}{h}=\lim _{h \rightarrow 0} \frac{2\left(\frac{2}{3} x+\frac{1}{3} h-2\right)}{h /}$
$=\lim _{h \rightarrow 0}\left(\frac{2}{3} x+\frac{1}{3} h-2\right)=\frac{2}{3} x+\frac{1}{3}(0)-2=\frac{2}{3} x-2=g^{\prime}(x)$

$$
g^{\prime}(3)=\frac{2}{3}(3)-2=2-2=0
$$

Example 5: Suppose that $f(x)=\sqrt{x^{2}+1}$. Find the equation of the tangent line at the point $f(2)=$

$$
\begin{aligned}
& f^{\prime}(x)=\lim _{h \rightarrow 0} \frac{f(x+h)-f(x)}{h}=\lim _{h \rightarrow 0} \frac{\sqrt{(x+h)^{2}+1}-\sqrt{x^{2}+1}}{h} \\
& =\lim _{h \rightarrow 0} \frac{\sqrt{(x+h)^{2}+1}-\sqrt{x^{2}+1}}{h} \cdot \frac{\sqrt{(x+h)^{2}+1}+\sqrt{x^{2}+1}}{\sqrt{(x+h)^{2}+1}+\sqrt{x^{2}+1}} \\
& =\lim _{h \rightarrow 0} \frac{(x+h)^{2}+1-\left(x^{2}+1\right)}{h\left(\sqrt{(x+h)^{2}+1}+\sqrt{\left.x^{2}+1\right)}\right.}=\lim _{h \rightarrow 0} \frac{x^{2}+2 x h+h^{2}+h-x^{2}-h}{h\left(\sqrt{(x+h)^{2}+1}+\sqrt{x^{2}+1}\right.} \\
& =\lim _{h \rightarrow 0} \frac{h((2 x+h)}{h\left(\sqrt{(x+h)^{2}+1}+\sqrt{x^{2}+1}\right)}=\lim _{h \rightarrow 0} \frac{2 x+h}{\sqrt{(x+h)^{2}+1}+\sqrt{x^{2}+1}} \\
& =\frac{2 x+0}{\sqrt{(x+0)^{2}+1}+\sqrt{x^{2}+1}}=\frac{2 x}{2 \sqrt{x^{2}+1}}=\frac{x}{\sqrt{x^{2}+1}}
\end{aligned}
$$

$$
\sqrt{2^{2}+1}
$$

$$
=\sqrt{5}
$$

$$
\text { Slope }=f^{\prime}(2)=\frac{2}{\sqrt{2^{2}+1}}=\frac{2}{\sqrt{5}}=\frac{2 \sqrt{5}}{5}
$$

$$
y-\sqrt{5}=\frac{2 \sqrt{5}}{5}(x-2)
$$

Example 6: Determine the equation of the tangent line to $f(x)=\frac{x-2}{x^{2}+1}$ at the point $\left(-2,-\frac{4}{5}\right)$.

$$
\begin{aligned}
& f^{\prime}(x)=\lim _{h \rightarrow 0} \frac{f(x+h)-f(x)}{h}=\lim _{h \rightarrow 0} \frac{\frac{x+h-2}{(x+h)^{2}+1}-\frac{x-2}{x^{2}+1}}{h} \\
& =\lim _{h \rightarrow 0} \frac{1}{h} \cdot\left[\frac{x+h-2}{(x+h)^{2}+1} \cdot \frac{x^{2}+1}{x^{2}+1}-\frac{x-2}{x^{2}+1} \cdot \frac{(x+h)^{2}+1}{x^{2}+(x+h)^{2}+1}\right] x^{2}+2-2 h+h^{2} \\
& =\lim _{h \rightarrow 0} \frac{1}{h} \cdot\left[\frac{x^{3}+x+h x^{2}+h-2 x^{2}-2-\left[x(x+h)^{2}+x-2(x+h)^{2}-2\right)}{\left((x+h)^{2}+1\right)\left(x^{2}+1\right)}\right] \\
& =\lim _{h \rightarrow 0} \frac{1}{h}\left[\frac{x^{3}+x+h x^{2}+h-22 x^{2}-2-x^{3}-2 x^{2} h-x h^{2}-x x+2-x^{2}+4 x h+2 h^{2}+2}{\left((x+h)^{2}+1\right)\left(x^{2}+1\right)}\right] \\
& =\lim _{h \rightarrow 0} \frac{1}{h}\left[\frac{h-x^{2} h-x h^{2}+4 x h+2 h^{2}}{\left((x+h)^{2}+1\right)\left(x^{2}+1\right)}\right] \\
& =\lim _{h \rightarrow 0} \frac{1}{k}\left[\frac{k\left(1-x^{2}-x h+4 x+2 h\right.}{\left((x+h)^{2}+1\right)\left(x^{2}+1\right)}\right]=\frac{1-x^{2}-x(0)+4 x+2(0)}{\left(x^{2}+1\right)^{2}} \\
& =\frac{1-x^{2}+4 x}{\left(x^{2}+1\right)^{2}} \text { still need en } \text { of tanguy } \text { in. }
\end{aligned}
$$

## Summary:

The slope of the secant line between two points is often called a difference quotient. The difference quotient of $f$ at $a$ can be written in either of the forms below.

$$
\frac{f(x)-f(a)}{x-a} \quad \frac{f(a+h)-f(a)}{h}
$$

Both of these give the slope of the secant line between two points: $(x, f(x))$ and $(a, f(a))$ or, alternatively, $(a, f(a))$ and $(a+h, f(a+h))$.
The slope of the secant line is also the average rate of change of $f$ between the two points.

The derivative of $f$ at $a$ is:

1) the limit of the slopes of the secant lines as the second point approaches the point $(a, f(a))$.
2) the slope of the tangent line to the curve $y=f(x)$ at the point where $x=a$.

3) the (instantaneous) rate of change of f with respect to $x$ at $a$.
4) $\lim _{x \rightarrow a} \frac{f(x)-f(a)}{x-a}$ (limit of the difference quotient)
5) $\lim _{h \rightarrow 0} \frac{f(a+h)-f(a)}{h}$ (limit of the difference quotient)


Common notations for the derivative of $y=f(x)$ :


$$
f^{\prime}(a) \text { or }\left.\frac{d y}{d x}\right|_{x=a}
$$

## Differentiability:

Definition: A function $f$ is differentiable at $a$ if $f^{\prime}(a)$ exists. It is differentiable on an open interval if it is differentiable at every number in the interval.

Theorem: If $f$ is differentiable at $a$, then $f$ is continuous at $a$.

Note: The converse is not true-there are functions that are continuous at a number but not differentiable.

Note: Open intervals: $(a, b),(-\infty, a),(a, \infty),(-\infty, \infty)$.

Closed intervals: $[a, b],(-\infty, a],[a, \infty),(-\infty, \infty)$.
To discuss differentiability on a closed interval, we need the concept of a one-sided derivative.

Derivative from the left: $\lim _{x \rightarrow a^{-}} \frac{f(x)-f(a)}{x-a}$
Derivative from the right: $\lim _{x \rightarrow a^{+}} \frac{f(x)-f(a)}{x-a}$
For a function $f$ to be differentiable on the closed interval [ $a, b$ ], it must be differentiable on the open interval $(a, b)$. In addition, the derivative from the right at $a$ must exist, and the derivative from the left at $b$ must exist.

Ways in which a function can fail to be differentiable:

1. Sharp corner
2. Cusp
3. Vertical tangent
4. Discontinuity


Example 8: $\quad$ Sketch the graph of a function for which $f(0)=2, f^{\prime}(0)=-1, f(2)=1$, $f^{\prime}(2)=\frac{1}{3}, f^{\prime}(3)>f^{\prime}(2)$, and $f^{\prime}(5)<0$.


Example 9: Use the graph of the function to draw the graph of the derivative.


Example 10: Use the graph of the function to draw the graph of the derivative.


Ex 4:

$$
g(x)=\frac{x^{2}-6 x}{3}
$$

(using the other
formula)

$$
\begin{aligned}
& =\frac{1}{3} \lim _{\omega \rightarrow x} \frac{\omega^{2}-6 \omega-x^{2}+6 x}{\omega-x}=\frac{1}{3} \lim _{\omega \rightarrow x} \frac{\omega^{2}-x^{2}+6 x-6 \omega}{\omega-x} \\
& =\frac{1}{3} \lim _{\omega \rightarrow x} \frac{(\omega+x)(\omega-x)-6(-x+\omega)}{\omega-x}=\frac{1}{3} \lim _{\omega \rightarrow x} \frac{(\omega-x)(\omega+x-6)}{\omega-1 x} \\
& =\frac{1}{3} \lim _{\omega \rightarrow x}(\omega+x-6)=\frac{1}{3}(x+x-6) \\
& =\frac{1}{3}(2 x-6)=\frac{2}{3} x-2
\end{aligned}
$$

