2.2: Basic Differentiation Rules and Rates of Change

Basic differentiation formulas:

1. $\frac{d}{dx}(c) = 0$ for any constant c.

2. $\frac{d}{dx}(x^n) = nx^{n-1}$ for any real number n. (Power Rule)

3. $\frac{d}{dx}[cf(x)] = c\frac{d}{dx}[f(x)]$

4. $\frac{d}{dx}[f(x)+g(x)] = \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)]$

5. $\frac{d}{dx}[f(x)-g(x)] = \frac{d}{dx}[f(x)] - \frac{d}{dx}[g(x)]$

Example 1: Find the derivative of $f(x) = 7$.

$\boxed{f'(x) = 0}$

Rule #1

Rule #2

Ex $1\frac{1}{2}$: $f(x) = x^4$

$\boxed{f'(x) = 4x^3}$

$f'(x) = 4x^{4-1}$

Ex $1\frac{2}{3}$: Rule #2 and Rule #3

$f(x) = 3x^6$

$f'(x) = 3(6x^5)$

$= \boxed{18x^5}$

$f'(x) = 3\frac{d}{dx}(x^6)$

$= 3(6x^5) = 18x^5$

Example 2: Find the derivative of $f(x) = 5x^3 - x^7 + 12x$.

$f'(x) = 15x^{3-1} - 7x^{7-1} + 12x^{1-1}$

$= 15x^2 - 7x^6 + 12x^0$

$= 15x^2 - 7x^6 + 12(1)$

$= \boxed{15x^2 - 7x^6 + 12}$

Example 3: Find the derivative of $g(x) = x^{17} + x^{3/2}$.

$g(x) = x^{17} + x^{3/2}$

$g'(x) = 17x^{16} + \frac{3}{2}x^{\frac{3}{2}-1}$

$= \boxed{17x^{16} + \frac{3}{2}x^{\frac{1}{2}}}$

Recall:

$$\sqrt[n]{x} = x^{\frac{1}{n}}$$

$$\frac{1}{x^n} = x^{-n}$$

$-\frac{1}{5} - 1 =$

$\frac{1}{5} - \frac{5}{5} = -\frac{4}{5}$

Example 4: Find the derivative of $f(x) = \sqrt[5]{x} + \frac{1}{x^2}$.

Rewrite: $f(x) = x^{\frac{1}{5}} + x^{-2}$

$f'(x) = \frac{1}{5}x^{-4/5} - 2x^{-3} = \boxed{\frac{1}{5\sqrt[5]{x^4}} - \frac{2}{x^3}}$

Example 5: Find the derivative of $f(x) = \frac{2}{\sqrt[4]{x}}$.

$f(x) = 2x^{-\frac{1}{4}}$

$f'(x) = 2\left(-\frac{1}{4}\right)x^{-\frac{1}{4}-1} = \boxed{-\frac{1}{2}x^{-\frac{5}{4}}} = \boxed{-\frac{1}{2\sqrt[4]{x^5}}}$

Example 6: Find the derivative of $h(x) = \left(\sqrt{x}\right)^5$.

$h(x) = \left(x^{\frac{1}{2}}\right)^5 = x^{5/2}$

$h'(x) = \frac{5}{2}x^{\frac{5}{2}-1} = \boxed{\frac{5}{2}x^{3/2}}$

Example 7: Find the derivative of $f(x) = -\sqrt[3]{6x^4}$.

$f(x) = -\sqrt[3]{6}\sqrt[3]{x^4} = -\sqrt[3]{6}\,x^{4/3}$

$f'(x) = -\sqrt[3]{6}\left(\frac{4}{3}\right)x^{\frac{4}{3}-1} = -\frac{4\sqrt[3]{6}}{3}x^{1/3}$

$= \boxed{-\frac{4\sqrt[3]{6x}}{3}}$

Example 8: Find the derivative of $f(x) = \frac{10}{x^4}$.

$f(x) = 10x^{-4}$

$f'(x) = \boxed{-40x^{-5}} = \boxed{-\frac{40}{x^5}}$

Example 9: Find the derivative of $g(x) = \dfrac{2\sqrt{x}}{7}$.

$$g(x) = \frac{2}{7}x^{1/2}$$

$$g'(x) = \frac{2}{7}\cdot\frac{1}{2}x^{\frac{1}{2}-1} = \frac{2}{14}x^{\frac{1}{2}-\frac{2}{2}} = \frac{1}{7}x^{-\frac{1}{2}} = \frac{1}{7x^{1/2}}$$

$$= \boxed{\frac{1}{7\sqrt{x}}}$$

Example 10: Find the derivative of $f(t) = \dfrac{3}{4t^2} - \sqrt[3]{7t}$.

$$f(t) = \frac{3}{4}t^{-2} - (7t)^{1/3}$$

$$= \frac{3}{4}t^{-2} - 7^{\frac{1}{3}}t^{\frac{1}{3}}$$

$$f'(t) = \frac{3}{4}\left(-2t^{-3}\right) - \sqrt[3]{7}\left(\frac{1}{3}t^{\frac{1}{3}-1}\right)$$

$$= \frac{3}{\cancel{4}_2}\cdot\frac{-\cancel{2}^1}{t^3} - \frac{\sqrt[3]{7}}{3}t^{-2/3}$$

$$= \boxed{-\frac{3}{2t^3} - \frac{\sqrt[3]{7}}{3t^{2/3}}}$$

$$= -\frac{3}{2t^3} - \frac{\sqrt[3]{7}}{3\sqrt[3]{t^2}}$$

$$= -\frac{3}{2t^3} - \frac{1}{3}\sqrt[3]{\frac{7}{t^2}}$$

Example 11: Find the derivative of $f(u) = \dfrac{7u^5 + u^2 - 9\sqrt{u}}{u^2}$.

$$f(u) = \frac{7u^5}{u^2} + \frac{u^2}{u^2} - \frac{9u^{1/2}}{u^2}$$

$$= 7u^3 + 1 - 9u^{\frac{1}{2}-2}$$

$$= 7u^3 + 1 - 9u^{-3/2}$$

$$f'(u) = 21u^2 + 0 - 9\left(-\frac{3}{2}u^{-\frac{3}{2}-1}\right)$$

$$= \boxed{21u^2 + \frac{27}{2}u^{-5/2}} = 21u^2 + \frac{27}{2\sqrt{u^5}}$$

Note: derivative of a constant is 0.
If $y = k$, then $y = kx^0$
so $y' = 0\cdot kx^{-1} = 0$

Example 12: Find the equation of the tangent line to the graph of $f(x) = 3x - x^2$ at the point $(-2, -10)$.

Check: $f(-2) = 3(-2) - (-2)^2$
$= -6 - 4 = -10$

$f'(x) = 3 - 2x$

Slope: $f'(-2) = 3 - 2(-2)$
$= 3 + 4 = 7$

Note: $\dfrac{d}{dx}(kx) = k(1x^0)$
$= k(1)$
$= k$

$$y - y_1 = m(x - x_1)$$

$$y - (-10) = 7(x - (-2))$$

$$y + 10 = 7(x + 2)$$

$$y = 7x + 14 - 10$$

$$\boxed{y = 7x + 4}$$

Example 13: Find the point(s) on the graph of $f(x) = x^2 + 6x$ where the tangent line is horizontal. Slope of a horizontal line is 0, so set $f'(x) = 0$.

$f'(x) = 2x + 6$

Set $f'(x) = 0$: $2x + 6 = 0$

$2x = -6$

$x = -3$

Find y-value: $f(-3) = (-3)^2 + 6(-3)$

$= 9 - 18 = -9$

$\boxed{\text{It is horizontal at the point } (-3, -9).}$

Definition: The *normal line* to a curve at the point *P* is defined to be the line passing through *P* that is perpendicular to the tangent line at that point.
(orthogonal)

$\frac{d}{dx}(x^n) = nx^{n-1}$

Example 14: Determine the equation of the normal line to the curve $y = \frac{1}{x}$ at the point $\left(3, \frac{1}{3}\right)$.

$y = x^{-1}$

$y' = \frac{dy}{dx} = \frac{d}{dx}\left(x^{-1}\right) = -1x^{-2} = -\frac{1}{x^2}$

$\frac{d}{dx}$ means take the derivative "with respect to x"

Slope of tangent line $= \left.\frac{dy}{dx}\right|_{x=3} = \left.-\frac{1}{x^2}\right|_{x=3} = -\frac{1}{(3)^2} = -\frac{1}{9}$

"evaluated at"

Slope of normal line is $+\frac{9}{1} = 9$.

(slopes of perpendicular lines are opposite reciprocals of each other)

Eqn of line: $y - y_1 = m(x - x_1)$

$y - \frac{1}{3} = 9(x - 3)$

$y - \frac{1}{3} = 9x - 27$

$y = 9x - \frac{81}{3} + \frac{1}{3}$

$\boxed{y = 9x - \frac{80}{3}}$

Derivatives of trigonometric functions:

$\frac{d}{dx}(\sin x) = \cos x$	$\frac{d}{dx}(\csc x) = -\csc x \cot x$
$\frac{d}{dx}(\cos x) = -\sin x$	$\frac{d}{dx}(\sec x) = \sec x \tan x$
$\frac{d}{dx}(\tan x) = \sec^2 x$	$\frac{d}{dx}(\cot x) = -\csc^2 x$

Notice: Derivatives of all the <u>co</u>functions have a minus sign.

Example 15: Find the derivative of $y = 2\cos x - 4\tan x$.

$$\frac{dy}{dx} = 2(-\sin x) - 4(\sec^2 x) = \boxed{-2\sin x - 4\sec^2 x}$$

Example 16: Find the derivative of $y = \frac{\sin x}{4} + 3x^4 + \pi^2$.

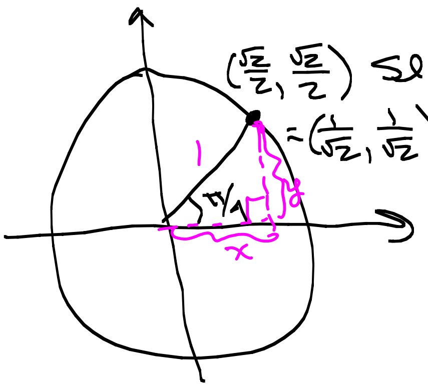
$$y = \frac{1}{4}\sin x + 3x^4 + \pi^2$$

$$\frac{dy}{dx} = \frac{1}{4}\cos x + 12x^3 + 0$$

$$= \boxed{\frac{1}{4}\cos x + 12x^3}$$

Example 17: Determine the equation of the tangent line to the graph of $y = \sec x$ at the point where $x = \frac{\pi}{4}$.

$$\frac{dy}{dx} = \frac{d}{dx}(\sec x) = \sec x \tan x$$



Slope: $m = \left.\frac{dy}{dx}\right|_{x=\frac{\pi}{4}} = \sec\left(\frac{\pi}{4}\right)\tan\left(\frac{\pi}{4}\right)$

$= \sqrt{2}\,(1) \Rightarrow \frac{\sin\frac{\pi}{4}}{\cos\frac{\pi}{4}} = \frac{\sqrt{2}/2}{\sqrt{2}/2} = 1$

$= \sqrt{2}$ reciprocal of $\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

Need the point:

Find y-value: $y\big|_{x=\pi/4} = \sec x\big|_{\pi/4} = \sec\frac{\pi}{4} = \sqrt{2}$

$y - y_1 = m(x - x_1)$

$$y - \sqrt{2} = \sqrt{2}\left(x - \frac{\pi}{4}\right) \Rightarrow \boxed{y = \sqrt{2}x - \frac{\pi\sqrt{2}}{4} + \sqrt{2}}$$

Example 18: Find the points on the curve $y = \tan x - 2x$ where the tangent line is horizontal.

Find $\frac{dy}{dx}$ and set it equal to 0.

$$\frac{dy}{dx} = \frac{d}{dx}(\tan x - 2x) = \sec^2 x - 2$$

Set $\frac{dy}{dx} = 0$: $\sec^2 x - 2 = 0$

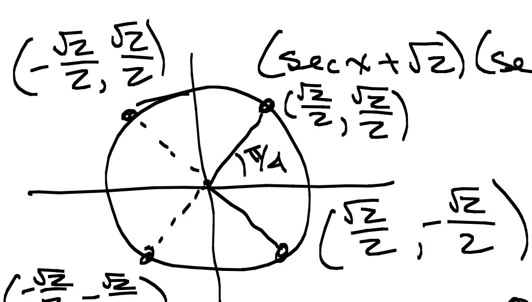
$\sec^2 x = 2$

OR $(\sec x + \sqrt{2})(\sec x - \sqrt{2}) = 0$

$\sec x = \pm\sqrt{2}$

$\cos x = \pm\frac{1}{\sqrt{2}}$

$\cos x = \pm\frac{\sqrt{2}}{2}$

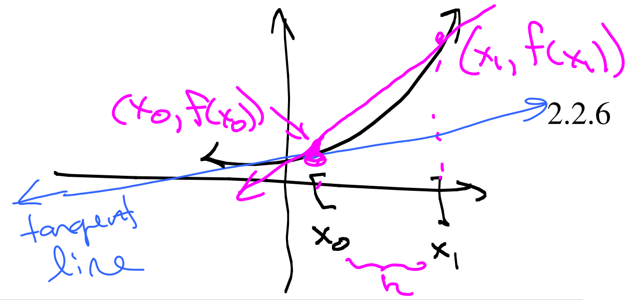


In $[0, 2\pi]$, $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

On $(-\infty, \infty)$, $\boxed{x = \frac{\pi}{4} + k\pi,\ x = \frac{3\pi}{4} + k\pi, \text{ where } k \text{ is any integer}}$

More compact form:

$x = \frac{\pi}{4} + \frac{k\pi}{2}$, where k is any integer



The derivative as a rate of change:

The average rate of change of $y = f(x)$ with respect to x over the interval $[x_0, x_1]$ is

$$\frac{\Delta y}{\Delta x} = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{f(x_0 + h) - f(x_0)}{h}, \text{ where } h = x_1 - x_0 \neq 0.$$

This is the same as the slope of the secant line joining points $P(x_0, f(x_0))$ and $Q(x_1, f(x_1))$.

The instantaneous rate of change (or, equivalently, just the rate of change) of f when $x = a$ is the slope of the tangent line to graph of f at the point $(a, f(a))$.

Therefore, the instantaneous rate of change is given by the derivative f'.

Example 19: Find the average rate of change in volume of a sphere with respect to its radius r as r changes from 3 to 4. Find the instantaneous rate of change when the radius is 3.

Average rate of change $= \dfrac{\Delta V}{\Delta r} = \dfrac{V(4) - V(3)}{4 - 3}$

$$= \frac{\frac{4}{3}\pi(4)^3 - \frac{4}{3}\pi(3)^3}{1} = \frac{4}{3}(64)\pi - \frac{4}{3}(27\pi) = \frac{4}{3}\pi(64 - 27) = \frac{4}{3}\pi(37) = \boxed{\frac{148\pi}{3}}$$

avg. rate of change

$$\begin{array}{r} 64 \\ -27 \\ \hline 37 \\ 4 \\ \hline 148 \end{array}$$

Volume

$V(r) = \frac{4}{3}\pi r^3$

Instantaneous rate of change:

$V(r) = \frac{4}{3}\pi r^3$

$$\frac{dV}{dr} = V'(r) = \frac{4}{3}\pi(3r^2) = \frac{12\pi r^2}{3} = 4\pi r^2 \Rightarrow V'(3) = 4\pi(3)^2 = \boxed{36\pi}$$

Example 20: Find the rate of change of the area of a circle with respect to (a) the diameter; (b) the circumference.

A = Area of Circle $= \pi r^2$

(a) diameter: $d = 2r$

$\frac{d}{2} = r$

$A = \pi r^2$ becomes $A(d) = \pi\left(\underbrace{\frac{d}{2}}_{r}\right)^2 = \frac{\pi d^2}{4} = \frac{\pi}{4} d^2$

$$A'(d) = \frac{\pi}{4}(2d) = \frac{2\pi d}{4} = \boxed{\frac{\pi d}{2}}$$

(b) Circumference:

$\underbrace{C = 2\pi r}$ $= \pi d$

$\frac{C}{2\pi} = r$

Area $= \pi r^2 = \pi\left(\frac{C}{2\pi}\right)^2$

$$A(C) = \pi\left(\frac{C^2}{4\pi^2}\right) = \frac{C^2}{4\pi} = \frac{1}{4\pi}(C^2)$$

$$A'(C) = \frac{1}{4\pi}(2C) = \frac{2C}{4\pi} = \boxed{\frac{C}{2\pi}}$$

Velocity:

If the independent variable represents *time*, then the derivative can be used to analyze motion.

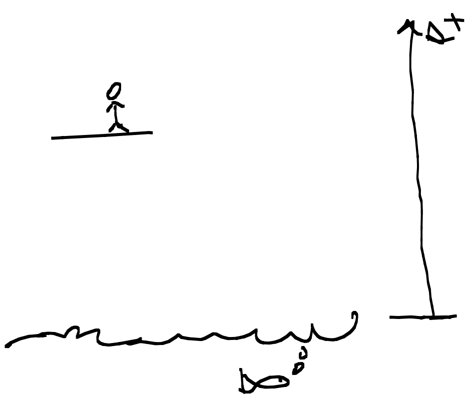
If the function $s(t)$ represents the position of an object, then the derivative $s'(t) = \frac{ds}{dt}$ is the velocity of the object.

(The velocity is the instantaneous rate of change in distance. The average velocity is the average rate of change in distance.)

Example 21: A person stands on a bridge 40 feet above a river. He throws a ball vertically upward with an initial velocity of 50 ft/sec. Its height (in feet) above the river after t seconds is $s = -16t^2 + 50t + 40$.

take derivative

a) What is the velocity after 3 seconds?
b) How high will it go?
c) How long will it take to reach a velocity of 20 ft/sec?
d) When will it hit the water? How fast will it be going when it gets there?



$v(t) = \frac{ds}{dt} = s'(t) = -32t + 50$

a) $s'(3) = -32(3) + 50 = -96 + 50$
$= \boxed{-46 \text{ ft/sec}}$
so it is going down

b) at maximum height, velocity $= s'(t) = 0$.
Set $s'(t) = 0$: $-32t + 50 = 0$
$50 = 32t$
$\frac{50}{32} = t$
$t = \frac{25}{16}$

At $t = \frac{25}{16}$ seconds, it's at max height.
$s\left(\frac{25}{16}\right) = -16\left(\frac{25}{16}\right)^2 + 50\left(\frac{25}{16}\right) + 4$
$= \boxed{79.0625 \text{ ft}}$

c) Set $s'(t) = 20$:
$-32t + 50 = 20$
$-32t = -30$
$t = \frac{-30}{-32} = \frac{15}{16}$
$\boxed{t = \frac{15}{16} \text{ sec}}$

d) When it reaches the water, $s(t) = 0 = -16t^2 + 50t + 40$
$t = \frac{-50 \pm \sqrt{(50)^2 - 4(-16)(40)}}{2(-16)} \Rightarrow \boxed{t \approx 3.785 \text{ sec}}$
$t \approx -0.66$ sec

How fast is it going?

Approx: $s'(3.785) = -32(3.785)+50$ ← Plug into $s'(t) = -32t+50$

$= \boxed{-71.12 \text{ ft/sec}}$

Example 22: Suppose a bullet is shot straight up at an initial velocity of 73 feet per second. If air resistance is neglected, its height from the ground (in feet) after t seconds is given by $h(t) = -16.1t^2 + 73t$.

a. The velocity after 2 seconds.
b. How high will the bullet go?
c. When will the bullet reach the ground?
d. How fast will it be traveling when it hits the ground?

Example 23: Suppose the position of a particle is given by $f(t) = t^4 - 32t + 7$. What is the velocity after 3 seconds? When is the particle at rest?