2.3: Product and Quotient Rules and Higher Order Derivatives

The Product Rule:

$$\frac{d}{dx}\left[f(x)g(x)\right] = f(x)g'(x) + g(x)f'(x)$$

“the first times the derivative of the second plus the second times the derivative of the first”

Example 1: Find the derivative of $f(x) = x^7(4x^3)$.

Example 2: Find the derivative of $f(x) = (4x^3 + x^2 - 2)(x^4 + 8)$.

Example 3: Find the derivative of $f(x) = \sqrt{x}(x^5 - 3x^2 + 12x)$.

Example 4: Find the derivative of $(4x^3 + 1)(\sqrt{x} + \frac{1}{x} - 2x)$.

<u>The Quotient Rule</u>:

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{\left[g(x)\right]^2}$$

<u>Example 5:</u> Find the derivative of $f(x) = \dfrac{-x^5}{4x^3}$.

<u>Example 6:</u> Find the derivative of $g(x) = \dfrac{4 - 2x^3}{x^4 + 3x^7}$.

<u>Example 7:</u> Find the derivative of $f(x) = \dfrac{\sqrt{x}}{x^3 - x^4}$.

Example 8: Find the derivative of $f(x) = x^2 \sin x$.

Example 9: Find the derivative of $f(x) = x + \sin x \cos x$.

Example 10: Find the derivative of $y = \dfrac{1 - \cos x}{\sin x}$.

Example 11: Prove that $\dfrac{d}{dx}(\csc x) = -\csc x \cot x$.

Higher order derivatives:

Once the derivative of $f(x)$ if also a function, it is possible to find the derivative of $f'(x)$ too. This is called the *second derivative* and is denoted $f''(x)$. The second derivative gives the instantaneous rate of change of the derivative. In other words, it tells us how fast the slope is changing.

Similarly, the derivative of $f''(x)$ can be calculated and this is called the *third derivative* $f'''(x)$.

In general, we can keep calculating the derivative of the previous derivative. The *nth derivative* is found by taking the derivative *n* times. The *nth derivative of f* is denoted $f^{(n)}(x)$.

Other notation: $y', y'', y''', \ldots, y^{(n)}$

$$\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \ldots, \frac{d^ny}{dx^n}$$

$$D'y, D^2y, D^3y, \ldots, D^ny$$

<u>Example 12:</u> Suppose that $y = -7x^5 + 6x^4 - \dfrac{2}{x}$. Find y'''.

<u>Example 13:</u> Suppose that $f(x) = \sqrt[3]{x}$. Find the second and third derivatives.

Using derivatives to describe the motion of an object:

If the dependent variable *t* represents time, and the function $s(t)$ represents the position (distance from a particular point) of an object, then

- the <u>velocity</u> $v(t)$ is the first derivative $s'(t) = \frac{ds}{dt}$.
- the <u>acceleration</u> $a(t)$ is the second derivative $s''(t) = \frac{dv}{dt}$.
- the <u>jerk</u> $j(t)$ is the third derivative $s'''(t) = \frac{da}{dt}$.
- the <u>speed</u> is the absolute value of the velocity $|v(t)| = \left|\frac{ds}{dt}\right|$.

<u>Example 14:</u> The position (in feet) of an object is given by $s(t) = t^4 - 32t + 7$, with *t* measured in seconds. Find functions representing the velocity, acceleration, and jerk. Find the velocity and acceleration after 1, 2, and 3 seconds.