

## 2.3: Product and Quotient Rules and Higher Order Derivatives

The Product Rule:

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$$

"the first times the derivative of the second plus the second times the derivative of the first"

Example 1: Find the derivative of  $f(x) = x^7(4x^3)$ .

Product Rule: 
$$\begin{aligned} f'(x) &= x^7 \frac{d}{dx}(4x^3) + (4x^3) \frac{d}{dx}(x^7) \\ &= x^7(12x^2) + 4x^3(7x^6) \\ &= (2x^9 + 28x^9) \\ &= \boxed{40x^9} \end{aligned}$$

without Product Rule:

$$\begin{aligned} f(x) &= x^7(4x^3) = 4x^{10} \\ f'(x) &= 40x^9 \end{aligned}$$

Example 2: Find the derivative of  $f(x) = (4x^3 + x^2 - 2)(x^4 + 8)$ .

$$\begin{aligned} f'(x) &= (4x^3 + x^2 - 2) \frac{d}{dx}(x^4 + 8) + (x^4 + 8) \frac{d}{dx}(4x^3 + x^2 - 2) \\ &= (4x^3 + x^2 - 2)(4x^3) + (x^4 + 8)(12x^2 + 2x) \\ &= \underline{16x^6} + \underline{4x^5} - \underline{8x^3} + \underline{12x^6} + \underline{2x^5} + \underline{96x^2} + \underline{16x} \\ &= \boxed{28x^6 + 6x^5 - 8x^3 + 96x^2 + 16x} \end{aligned}$$

Example 3: Find the derivative of  $f(x) = \sqrt{x}(x^5 - 3x^2 + 12x)$ .

$$\begin{aligned} f(x) &= x^{\frac{1}{2}}(x^5 - 3x^2 + 12x) \\ f'(x) &= x^{\frac{1}{2}} \frac{d}{dx}(x^5 - 3x^2 + 12x) + (x^5 - 3x^2 + 12x) \frac{d}{dx}(x^{\frac{1}{2}}) \\ &= x^{\frac{1}{2}}(5x^4 - 6x + 12) + (x^5 - 3x^2 + 12x)\left(\frac{1}{2}x^{-\frac{1}{2}}\right) \\ &\quad \text{then simplify.} \end{aligned}$$

Example 4: Find the derivative of  $(4x^3 + 1)(\sqrt{x} + \frac{1}{x} - 2x)$ .

The Quotient Rule:

$$\frac{d}{dx} \left[ \frac{f(x)}{g(x)} \right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

Example 5: Find the derivative of  $f(x) = \frac{-x^5}{4x^3}$ .

Using Quotient Rule:

$$f'(x) = \frac{4x^3 \frac{d}{dx}(-x^5) - (-x^5) \frac{d}{dx}(4x^3)}{(4x^3)^2}$$

$$= \frac{4x^3(-5x^4) + x^5(12x^2)}{(16x^6)} = \frac{-20x^7 + 12x^7}{(16x^6)} = \frac{-8x^7}{(16x^6)} = -\frac{1}{2}x$$

Without Quotient Rule:

$$f(x) = \frac{-x^5}{4x^3} = -\frac{1}{4}x^2$$

$$f'(x) = -\frac{1}{4}(2x) = -\frac{1}{2}x$$

Same ↗

Example 6: Find the derivative of  $g(x) = \frac{4-2x^3}{x^4+3x^7}$ .

$$g'(x) = \frac{(x^4+3x^7) \frac{d}{dx}(4-2x^3) - (4-2x^3) \frac{d}{dx}(x^4+3x^7)}{(x^4+3x^7)^2}$$

$$= \frac{(x^4+3x^7)(-6x^2) - (4-2x^3)(4x^3+21x^6)}{(x^4+3x^7)^2}$$

$$= \frac{-6x^6 - 18x^9 - (16x^3 + 84x^6 - 8x^6 - 42x^9)}{(x^4+3x^7)^2}$$

$$= \frac{-6x^6 - 18x^9 - 16x^3 - 76x^6 + 42x^9}{(x^4+3x^7)^2}$$

Example 7: Find the derivative of  $f(x) = \frac{\sqrt{x}}{x^3-x^4}$ .

$$f'(x) = \frac{(x^3-x^4) \frac{d}{dx}(x^{\frac{1}{2}}) - x^{\frac{1}{2}} \frac{d}{dx}(x^3-x^4)}{(x^3-x^4)^2}$$

$$= \frac{(x^3-x^4)(\frac{1}{2}x^{-\frac{1}{2}}) - x^{\frac{1}{2}}(3x^2-4x^3)}{(x^3-x^4)^2}$$

$$= \frac{24x^9 - 82x^6 - 16x^3}{(x^4+3x^7)^2}$$

$$= \frac{2x^3(12x^6 - 41x^3 - 8)}{(x^4+3x^7)^2}$$

$$= \frac{2x^3(12x^6 - 41x^3 - 8)}{x^8(1+3x^3)^2}$$

$$= \frac{2(12x^6 - 41x^3 - 8)}{x^5(1+3x^3)^2}$$

**Example 8:** Find the derivative of  $f(x) = x^2 \sin x$ .

$$\begin{aligned} f'(x) &= x^2 \frac{d}{dx}(\sin x) + (\sin x) \frac{d}{dx}(x^2) \\ &= x^2(\cos x) + (\sin x)(2x) \\ &= \boxed{x^2 \cos x + 2x \sin x} = \boxed{x(x \cos x + 2 \sin x)} \end{aligned}$$

**Example 9:** Find the derivative of  $f(x) = x + \sin x \cos x$ .

$$\begin{aligned} f'(x) &= 1 + (\sin x) \frac{d}{dx}(\cos x) + (\cos x) \frac{d}{dx}(\sin x) \\ &= 1 + (\sin x)(-\sin x) + (\cos x)(\cos x) \\ &= \underbrace{1 - \sin^2 x}_{\cos^2 x} + \cos^2 x \\ &= \cos^2 x + \cos^2 x = \boxed{2 \cos^2 x} \end{aligned}$$

**Example 10:** Find the derivative of  $y = \frac{1 - \cos x}{\sin x}$ .

$$\begin{aligned} \frac{dy}{dx} &= y' = \frac{d}{dx}(y) = \frac{(\sin x) \frac{d}{dx}(1 - \cos x) - (1 - \cos x) \frac{d}{dx}(\sin x)}{(\sin x)^2} \\ &= \frac{(\sin x)(0 - (-\sin x)) + (-1 + \cos x)(\cos x)}{\sin^2 x} \\ &= \frac{(\sin x)(\sin x) - \cos x + \cos^2 x}{\sin^2 x} = \frac{\sin^2 x - \cos x + \cos^2 x}{\sin^2 x} \end{aligned}$$

**Example 11:** Prove that  $\frac{d}{dx}(\csc x) = -\csc x \cot x$ .

(Use the fact that

$\frac{d}{dx}(\cos x) = -\sin x$  and  $\frac{d}{dx}(\sin x) = \cos x$ , along with the quotient rule)

$$\frac{d}{dx}(\csc x) = \frac{d}{dx}\left(\frac{1}{\sin x}\right)$$

$$= \frac{\sin x \frac{d}{dx}(1) - 1 \frac{d}{dx}(\sin x)}{(\sin x)^2} = \frac{(\sin x)(0) - 1(-\cos x)}{\sin^2 x} = \frac{-\cos x}{\sin^2 x}$$

$$= -\frac{\cos x}{\sin x} \left(\frac{1}{\sin x}\right) = -\cot x \csc x = -\csc x \cot x$$

$$= \frac{1 - \cos x}{\sin^2 x} = \frac{1 - \cos x}{1 - \cos^2 x}$$

$$= \frac{1 - \cos x}{(1 + \cos x)(1 - \cos x)}$$

$$= \boxed{\frac{1}{1 + \cos x}}$$



### Higher order derivatives:

Once the derivative of  $f(x)$  is also a function, it is possible to find the derivative of  $f'(x)$  too. This is called the *second derivative* and is denoted  $f''(x)$ . The second derivative gives the instantaneous rate of change of the derivative. In other words, it tells us how fast the slope is changing.

Similarly, the derivative of  $f''(x)$  can be calculated and this is called the *third derivative*  $f'''(x)$ .

In general, we can keep calculating the derivative of the previous derivative. The  *$n$ th derivative* is found by taking the derivative  $n$  times. The  *$n$ th derivative* of  $f$  is denoted  $f^{(n)}(x)$ .

Other notation:  $y', y'', y''', \dots, y^{(n)}$

$$\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \dots, \frac{d^ny}{dx^n}$$

$$D'y, D^2y, D^3y, \dots, D^ny$$

*Note:*

$$\begin{aligned} \text{1st deriv. } \frac{d}{dx}(y) &= \frac{dy}{dx} \\ \text{2nd deriv. } \frac{d}{dx}\left(\frac{dy}{dx}\right) &= \frac{d^2y}{dx^2} \end{aligned}$$

**Example 12:** Suppose that  $y = -7x^5 + 6x^4 - \frac{2}{x}$ . Find  $y'''$ .

$$\begin{aligned} y &= -7x^5 + 6x^4 - 2x^{-1} \\ y' &= -35x^4 + 24x^3 + 2x^{-2} \\ y'' &= -140x^3 + 72x^2 - 4x^{-3} \\ y''' &= -420x^2 + 144x + 12x^{-4} \end{aligned} = \boxed{-420x^2 + 144x + \frac{12}{x^4}}$$

**Example 13:** Suppose that  $f(x) = \sqrt[3]{x}$ . Find the second and third derivatives.

$$\begin{aligned} f(x) &= x^{1/3} \\ f'(x) &= \frac{1}{3}x^{-2/3} \\ f''(x) &= \frac{1}{3}\left(-\frac{2}{3}x^{-5/3}\right) = \boxed{-\frac{2}{9}x^{-5/3}} = \boxed{-\frac{2}{9}\sqrt[3]{x^5}} \\ f'''(x) &= -\frac{2}{9}\left(-\frac{5}{3}x^{-8/3}\right) = \boxed{\frac{10}{27}x^{-8/3}} = \boxed{\frac{10}{27}\sqrt[3]{x^8}} \end{aligned}$$

### Using derivatives to describe the motion of an object:

If the dependent variable  $t$  represents time, and the function  $s(t)$  represents the position (distance from a particular point) of an object, then

- the velocity  $v(t)$  is the first derivative  $s'(t) = \frac{ds}{dt}$ .
- the acceleration  $a(t)$  is the second derivative  $s''(t) = \frac{dv}{dt}$ .
- the jerk  $j(t)$  is the third derivative  $s'''(t) = \frac{da}{dt}$ .
- the speed is the absolute value of the velocity  $|v(t)| = \left| \frac{ds}{dt} \right|$ .

**Example 14:** The position (in feet) of an object is given by  $s(t) = t^4 - 32t + 7$ , with  $t$  measured in seconds. Find functions representing the velocity, acceleration, and jerk. Find the velocity and acceleration after 1, 2, and 3 seconds.

$$\text{Velocity: } s'(t) = 4t^3 - 32$$

$$\text{acceleration: } s''(t) = 12t^2$$

$$\text{jerk: } s'''(t) = 24t$$

$$\begin{aligned} s'(1) &= 4(1)^3 - 32 = -28 \text{ ft/sec} \\ s''(1) &= 12(1)^2 = 12(\text{ft/sec})/\text{sec} \\ &= 12 \text{ ft/sec/sec} \\ &= \frac{12 \text{ ft}}{\text{sec}} = \frac{12 \text{ ft}}{\text{sec}} \left( \frac{1}{\text{sec}} \right) \\ &= 12 \text{ ft/sec}^2 = 12 \text{ ft/s}^2 \end{aligned}$$

$$s'(2) = 4(2)^3 - 32 = 0 \text{ ft/sec}$$

$$s''(2) = 12(2)^2 = 48 \text{ ft/s}^2$$

$$s'(3) = 4(3)^3 - 32 = 76 \text{ ft/s}$$

$$s''(3) = 12(3)^2 = 108 \text{ ft/s}^2$$

