

2.4: The Chain Rule

The chain rule allows us to find the derivative of many more functions.

Note: if $y = x^2$,
then $\frac{dy}{dx} = 2x$

Example 1: Suppose $g(x) = (x^3 + 2)^2$. Find $g'(x)$.

One way: $g(x) = (x^3 + 2)(x^3 + 2)$

$$\begin{aligned} &= x^6 + 4x^3 + 4 \\ g'(x) &= \boxed{6x^5 + 12x^2} \\ &= 6x^2(x^3 + 2) \\ &= \boxed{6x^2(x^3 + 2)} \end{aligned}$$

Using Chain rule:

$$\begin{aligned} g(x) &= (x^3 + 2)^2 \\ g'(x) &= 2(x^3 + 2) \frac{d}{dx}(x^3 + 2) \\ &= 2(x^3 + 2)(3x^2) \end{aligned}$$

\times derivative of outside function, with respect to the inside function

The chain rule lets us differentiate composite functions.

The Chain Rule:

If f and g are both differentiable and $F(x) = f(g(x))$, then F is differentiable and

$$F'(x) = f'(g(x))g'(x).$$

Alternatively, if $y = f(u)$ and $u = f(x)$ are both differentiable functions, then

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}.$$

With the chain rule, we take the derivative of the “outer function” and multiply by the derivative of the “inner function”.

Example 2: Find the derivative of $h(x) = (x^3 + 2)^{50}$.

Let $u = x^3 + 2$

$y = u^{50}$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= 50u^{49}(3x^2)$$

$$= 50(x^3 + 2)^{49}(3x^2)$$

$$= \boxed{150x^2(x^3 + 2)^{49}}$$

$$\begin{aligned} h'(x) &= 50(x^3 + 2)^{49} \frac{d}{dx}(x^3 + 2) \\ &= 50(x^3 + 2)^{49}(3x^2) \\ &= \boxed{150x^2(x^3 + 2)^{49}} \end{aligned}$$

$\frac{dy}{dx}$ etc called Leibniz notation

Example 3: Find the derivative of $f(x) = \sqrt{2x^3 - 5x}$.

$$\begin{aligned} f(x) &= (2x^3 - 5x)^{1/2} \\ f'(x) &= \frac{1}{2}(2x^3 - 5x)^{-\frac{1}{2}} \frac{d}{dx}(2x^3 - 5x) \\ &= \frac{1}{2}(2x^3 - 5x)^{-\frac{1}{2}}(6x^2 - 5) \\ &= \boxed{\frac{6x^2 - 5}{2\sqrt{2x^3 - 5x}}} \end{aligned}$$

Example 4: Suppose that $y = \frac{2}{(3x^5 - 4)^3}$. Find $\frac{dy}{dx}$.

$$\begin{aligned} y &= 2(3x^5 - 4)^{-3} \\ \frac{dy}{dx} &= -6(3x^5 - 4)^{-4} \frac{d}{dx}(3x^5 - 4) \\ &= -6(3x^5 - 4)^{-4}(15x^4) = -90x^4(3x^5 - 4)^{-4} \end{aligned}$$

Example 5: Find the derivative of $f(x) = \sqrt[3]{\cos x}$.

$$\begin{aligned} f(x) &= (\cos x)^{1/3} \\ f'(x) &= \frac{1}{3}(\cos x)^{-\frac{2}{3}} \frac{d}{dx}(\cos x) \\ &= \frac{1}{3}(\cos x)^{-\frac{2}{3}}(-\sin x) = \boxed{-\frac{\sin x}{3(\cos x)^{2/3}}} \end{aligned}$$

Ex: $4^{\frac{1}{2}}$: $y = \frac{x^2}{2}$

$y = \frac{1}{2}x^4$
 $y' = \frac{1}{2}x^3 = 2x^3$

Example 6: Find the derivative of $g(x) = \cos \sqrt[3]{x}$.

$$\begin{aligned} g(x) &= \cos(\sqrt[3]{x}) = \cos(x^{1/3}) \\ g'(x) &= -\sin(x^{1/3}) \frac{d}{dx}(x^{1/3}) \\ &= -\sin(x^{1/3})(\frac{1}{3}x^{-2/3}) = \boxed{-\frac{\sin(x^{1/3})}{3x^{2/3}}} = -\frac{\sin \sqrt[3]{x}}{3\sqrt[3]{x^2}} \end{aligned}$$

↑ inverse-cosine

Example 7: Suppose that $y = \frac{1}{\cos x}$. Find $\frac{dy}{dx}$.

$$\begin{aligned} y &= (\cos x)^{-1} \\ \frac{dy}{dx} &= -1(\cos x)^{-2} \frac{d}{dx}(\cos x) \\ &= -1(\cos x)^{-2}(-\sin x) \\ &= \frac{\sin x}{(\cos x)^2} = \left(\frac{1}{\cos x}\right)\left(\frac{\sin x}{\cos x}\right) = \sec x \tan x \quad \checkmark \end{aligned}$$

* Not $\cos^{-1} x$

* Not $\cos x^{-1}$

Example 8: Suppose that $h(x) = (3x^2 - 4)^3(2x - 9)^2$. Find $h'(x)$.

Product Rule:
$$\begin{aligned} h'(x) &= (3x^2 - 4)^3 \frac{d}{dx}(2x - 9)^2 + (2x - 9)^2 \frac{d}{dx}(3x^2 - 4)^3 \\ &= (3x^2 - 4)^3(2(2x - 9)(2)) + (2x - 9)^2(-3(3x^2 - 4)^2(6x)) \\ &= \underline{4(3x^2 - 4)^3(2x - 9)} + \underline{18x(2x - 9)^2(3x^2 - 4)^2} \\ &= 2(3x^2 - 4)^2(2x - 9)[2(3x^2 - 4) + 9x(2x - 9)] \\ &= 2(3x^2 - 4)^2(2x - 9)[6x^2 - 8 + 18x^2 - 81x] \\ &= \boxed{2(3x^2 - 4)^2(2x - 9)(24x^2 - 81x - 8)} \end{aligned}$$

Example 9: Suppose that $f(x) = \left(\frac{2x+1}{2x-1}\right)^5$. Find $f'(x)$.

$$\begin{aligned} f'(x) &= 5\left(\frac{2x+1}{2x-1}\right)^4 \frac{d}{dx}\left(\frac{2x+1}{2x-1}\right) = 5\left(\frac{2x+1}{2x-1}\right)^4 \left(\frac{(2x-1)(2) - (2x+1)(2)}{(2x-1)^2}\right) \\ &= 5\left(\frac{2x+1}{2x-1}\right)^4 \left(\frac{4x-2-4x-2}{(2x-1)^2}\right) = 5\left(\frac{2x+1}{2x-1}\right)^4 \left(\frac{-4}{(2x-1)^2}\right) \\ &= \boxed{\frac{-20(2x+1)^4}{(2x-1)^6}} \end{aligned}$$

Example 10: Find the derivative of $y = \cos(\sin(\pi x^2))$.

$$\begin{aligned} \frac{dy}{dx} &= -\sin(\sin(\pi x^2)) \frac{d}{dx}(\sin(\pi x^2)) \\ &= -\sin(\sin(\pi x^2)) \cos(\pi x^2) \frac{d}{dx}(\pi x^2) \\ &= -\sin(\sin(\pi x^2)) (\cos(\pi x^2)) (2\pi x) \\ &= \boxed{-2\pi x \sin(\sin(\pi x^2)) (\cos(\pi x^2))} \end{aligned}$$

Example 11: Find the derivative of $g(x) = \frac{\cos^2 x}{\sqrt{4x+1}}$. $g(x) = \frac{(\cos x)^2}{(4x+1)^{1/2}}$

$$\begin{aligned} g'(x) &= \frac{(4x+1)^{1/2} \frac{d}{dx} (\cos x)^2 - (\cos x)^2 \frac{d}{dx} (4x+1)^{1/2}}{[(4x+1)^{1/2}]^2} \\ &= \frac{(4x+1)^{1/2} (2)(\cos x)(-\sin x) - (\cos x)^2 (\frac{1}{2})(4x+1)^{-1/2} (4)}{-(2\cos x \sin x)\sqrt{4x+1} - \frac{2\cos x^2}{2\sqrt{4x+1}} \cdot \frac{\sqrt{4x+1}}{\sqrt{4x+1}}} = \end{aligned}$$

Example 12: Find the first and second derivatives of $f(x) = (x^2 + 4)^5$.

$$\begin{aligned} f'(x) &= 5(x^2 + 4)^4 (2x) = \boxed{10x(x^2 + 4)^4} \\ f''(x) &= 10x \frac{d}{dx} (x^2 + 4)^4 + (x^2 + 4)^4 \frac{d}{dx} (10x) \\ &= 10x(4)(x^2 + 4)^3 (2x) + (x^2 + 4)^4 (10) \\ &= 80x^2(x^2 + 4)^3 + 10(x^2 + 4)^4 \\ &= 10(x^2 + 4)^3 [8x^2 + x^2 + 4] = \boxed{10(x^2 + 4)^3 (9x^2 + 4)} \end{aligned}$$

Example 13: Find the first and second derivatives of $f(x) = \cos(3x^2)$.

$$\begin{aligned} f'(x) &= -(\sin(3x^2))(6x) = \boxed{-6x \sin(3x^2)} \\ f''(x) &= -6x \frac{d}{dx} (\sin(3x^2)) + \sin(3x^2) \frac{d}{dx} (-6x) \\ &= -6x(\cos(3x^2))(6x) + (\sin(3x^2))(-6) \\ &= \boxed{-36x^2 \cos(3x^2) - 6 \sin(3x^2)} \end{aligned}$$

Example 14: Find the first and second derivatives of $y = \frac{2}{(3x+5)^2}$

$$\frac{dy}{dx} = y' = -4(3x+5)^{-3} \frac{d}{dx}(3x+5) = -4(3x+5)^{-3}(3) = \boxed{-12(3x+5)^{-3}}$$

$$= -\frac{12}{(3x+5)^3}$$

$$\frac{d^2y}{dx^2} = y'' = 36(3x+5)^{-4}(3)$$

$$= \boxed{108(3x+5)^{-4}} = \boxed{\frac{108}{(3x+5)^4}}$$

Example 15: Find the equation of the tangent line to $y = \frac{x^2}{\sin \pi x + 1}$ at the point where $x=1$.

Point: $y \Big|_{x=1} = \frac{x^2}{\sin \pi x + 1} \Big|_{x=1} = \frac{1^2}{\sin(\pi) + 1} = \frac{1}{0+1} = \frac{1}{1} = 1$

$$\frac{dy}{dx} = \frac{(\sin \pi x + 1) \frac{d}{dx}(x^2) - x^2 \frac{d}{dx}(\sin \pi x + 1)}{(\sin \pi x + 1)^2} = \frac{(\sin \pi x + 1)(2x) - x^2((\cos(\pi x))(\pi + 0))}{(\sin \pi x + 1)^2}$$

$$= \frac{2x(\sin \pi x + 1) - \pi x^2 \cos \pi x}{(\sin \pi x + 1)^2}$$

$$\frac{dy}{dx} \Big|_{x=1} = \frac{2(1)(\sin \pi + 1) - \pi(1)^2 \cos \pi}{(\sin \pi + 1)^2} = \frac{2(0+1) - \pi(-1)}{(0+1)^2}$$

$$= \frac{2+\pi}{1} = 2+\pi \quad \text{slope of tangent line}$$

Pt.: $(1, 1)$

$$y - y_1 = m(x - x_1)$$

$$y - 1 = (2+\pi)(x-1)$$

$$y - 1 = 2x - 2 + \pi x - \pi$$

$$y = \boxed{2x + \pi x - 1 - \pi}$$

$$y = (2+\pi)x - 1 - \pi$$

E_x II cont'd:

$$\frac{-2\cos x \sin \sqrt{4x+1} - \frac{2\sqrt{4x+1}\cos^2 x}{2\sqrt{4x+1}}}{\sqrt{4x+1}}$$

$$= \frac{-2(\cos x \sin x)(4x+1) - 2(\cos x)^2}{(4x+1)^{3/2}}$$

$$= \boxed{\frac{-2\cos x ((4x+1)\sin x - \cos x)}{(4x+1)^{3/2}}}$$