

2.5: Implicit Differentiation

Example 1: Given the equation $x^3 - 4y - 9x^2 = 5$, find $\frac{dy}{dx}$ by

- a) Solving explicitly for y.
- b) Implicit differentiation.

@ solve for y: $x^3 - 9x^2 - 5 = 4y$

$$y = \frac{x^3 - 9x^2 - 5}{4}$$

$$y = \frac{1}{4}(x^3 - 9x^2 - 5)$$

$$\frac{dy}{dx} = \frac{1}{4}(3x^2 - 18x) = \boxed{\frac{3}{4}x^2 - \frac{9}{2}x}$$

Example 2: Find $\frac{dy}{dx}$ for $xy = 4$.

$$\frac{d}{dx}(xy) = \frac{d}{dx}(4)$$

$$x \frac{dy}{dx} + y \frac{d}{dx}(x) = 0$$

$$x \frac{dy}{dx} + y(1) = 0$$

Solve for $\frac{dy}{dx}$:

$$x \frac{dy}{dx} = -y$$

⑥ Implicit differentiation:

$$x^3 - 4y - 9x^2 = 5$$

$$\frac{d}{dx}(x^3 - 4y - 9x^2) = \frac{d}{dx}(5)$$

$$3x^2 - 4 \frac{dy}{dx} - 18x = 0$$

Solve for $\frac{dy}{dx}$:

$$3x^2 - 18x = 4 \frac{dy}{dx}$$

$$\frac{3x^2}{4} - \frac{18x}{4} = \frac{4 \frac{dy}{dx}}{4}$$

$$\frac{dy}{dx} = \boxed{\frac{3x^2}{4} - \frac{9x}{2}}$$

Example 3: Find $\frac{dy}{dx}$ for the equation $x^3y - 2x^2y^3 + x^2 - 3 = 0$.

$$\frac{d}{dx}(x^3y - 2x^2y^3 + x^2 - 3) = \frac{d}{dx}(0)$$

$$x^3 \frac{dy}{dx} + y \frac{d}{dx}(x^3) - 2x^2 \frac{d}{dx}(y^3) + y^3 \frac{d}{dx}(-2x^2) + 2x - 0 = 0$$

$$x^3 \frac{dy}{dx} + y(3x^2) - 2x^2(3y^2 \frac{dy}{dx}) + y^3(-4x) + 2x = 0$$

$$x^3 \frac{dy}{dx} + 3x^2y - 6x^2y^2 \frac{dy}{dx} - 4xy^3 + 2x = 0$$

Solve for $\frac{dy}{dx}$. (Put terms with $\frac{dy}{dx}$ all on 1 side.)

Put terms without $\frac{dy}{dx}$ on other side.

$$x^3 \frac{dy}{dx} - 6x^2y^2 \frac{dy}{dx} = -3x^2y + 4xy^3 - 2x$$

$$\frac{dy}{dx} (x^3 - 6x^2y^2) = -3x^2y + 4xy^3 - 2x \implies \boxed{\frac{dy}{dx} = \frac{-3x^2y + 4xy^3 - 2x}{x^3 - 6x^2y^2}}$$

Example 4: Find $\frac{dy}{dx}$ for the equation $\frac{1}{y^4} - \frac{5}{x^4} = 1$.

Rewrite:

$$\begin{aligned} y^{-4} - 5x^{-4} &= 1 \\ \frac{d}{dx}(y^{-4} - 5x^{-4}) &= \frac{d}{dx}(1) \quad (1) \\ -4y^{-5} \frac{dy}{dx} + 20x^{-5} &= 0 \\ -\frac{4}{y^5} \frac{dy}{dx} &= -\frac{20}{x^5} \end{aligned}$$

$$-\frac{4}{y^5} \left(-\frac{4}{y^5} \frac{dy}{dx} \right) = -\frac{20}{x^5} \left(-\frac{4}{y^5} \right)$$

$$\frac{dy}{dx} = \frac{20y^5}{4x^5}$$

$$\frac{dy}{dx} = \frac{5y^5}{x^5}$$

Example 5: Find $\frac{dy}{dx}$ for the equation $(x-y)^4 = y^2$.

$$(x-y)^4 = y^2$$

$$\frac{d}{dx}(x-y)^4 = \frac{d}{dx}(y^2)$$

$$4(x-y)^3 \frac{d}{dx}(x-y) = 2y \frac{dy}{dx}$$

$$4(x-y)^3 \left(1 - \frac{dy}{dx}\right) = 2y \frac{dy}{dx}$$

$$4(x-y)^3 (1) - 4(x-y)^3 \frac{dy}{dx} = 2y \frac{dy}{dx}$$

move to right-hand side

$$4(x-y)^3 = 2y \frac{dy}{dx} + 4(x-y)^3 \frac{dy}{dx}$$

$$4(x-y)^3 = \frac{dy}{dx} (2y + 4(x-y)^3)$$

$$\frac{4(x-y)^3}{2y + 4(x-y)^3} = \frac{dy}{dx}$$

Example 6: Find $\frac{dy}{dx}$ for the equation $x + \cos(x^2y) = y$

$$\frac{d}{dx}(x + \cos(x^2y)) = \frac{d}{dx}(y)$$

$$1 - (\sin(x^2y)) \frac{d}{dx}(x^2y) = \frac{dy}{dx}$$

$$1 - (\sin(x^2y)) (x^2 \frac{dy}{dx} + y \cdot 2x) = \frac{dy}{dx}$$

$$1 - x^2 \frac{dy}{dx} \sin(x^2y) - 2xy \sin(x^2y) = \frac{dy}{dx}$$

$$1 - 2xy \sin(x^2y) = \frac{dy}{dx} + x^2 \frac{dy}{dx} \sin(x^2y)$$

$$(-2xy \sin(x^2y)) = \frac{dy}{dx} (1 + x^2 \sin(x^2y))$$

dividing out a -1 :

$$\frac{dy}{dx} = \frac{2(x-y)^3}{y+2(x-y)^3}$$

multiply by $\frac{-1}{-1}$

$$\frac{dy}{dx} = \frac{-1 + 2xy \sin(x^2y)}{-1 - x^2 \sin(x^2y)}$$

equivalent
(so both are correct)

$$\frac{1 - 2xy \sin(x^2y)}{1 + x^2 \sin(x^2y)} = \frac{dy}{dx}$$

Finding higher-order derivatives using implicit differentiation:

To find the second derivative, denoted $\frac{d^2y}{dx^2}$, differentiate the first derivative $\frac{dy}{dx}$ with respect to x .

$$\frac{d^2y}{dx^2} = \frac{d\left(\frac{dy}{dx}\right)}{dx} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

Example 7: Find $\frac{d^2y}{dx^2}$ for the equation $x^3 - 2x^2 = y$.

$$\frac{d}{dx}(x^3 - 2x^2) = \frac{d}{dx}(y)$$

$$3x^2 - 4x = \frac{dy}{dx}$$

$$\begin{aligned} \frac{d^2y}{dx^2} &= \frac{d}{dx}(3x^2 - 4x) \\ &= \boxed{6x - 4} \end{aligned}$$

Example 8: Find $\frac{d^2y}{dx^2}$ for the equation $xy^2 - y = 3$.

$$\frac{d}{dx}(xy^2 - y) = \frac{d}{dx}(3)$$

$$x \frac{d}{dx}(y^2) + y^2 \frac{d}{dx}(x) - \frac{dy}{dx} = 0$$

$$x\left(2y \frac{dy}{dx}\right) + y^2(1) - \frac{dy}{dx} = 0$$

$$2xy \frac{dy}{dx} + y^2 - \frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(2xy - 1) = -y^2$$

$$\frac{dy}{dx} = \frac{-y^2}{2xy - 1} = \frac{y^2}{-2xy + 1}$$

$$\begin{aligned} \frac{d^2y}{dx^2} &= \\ &= \frac{(2xy-1)\frac{d}{dx}(-y^2) - (-y^2)\frac{d}{dx}(2xy-1)}{(2xy-1)^2} \\ &= \frac{(2xy-1)(-2y \frac{dy}{dx}) + y^2(2x \frac{dy}{dx} + y(2y-1) \cancel{+ 0})}{(2xy-1)^2} \\ &= \frac{-4xy^2 \frac{dy}{dx} + 2y \frac{dy}{dx} + 2xy^2 \frac{dy}{dx} + 2y^3}{(2xy-1)^2} \\ &= \frac{-2xy^2 \frac{dy}{dx} + 2y \frac{dy}{dx} + 2y^3}{(2xy-1)^2} \end{aligned}$$

Continued at end of notes.

$$x=2, y=\sqrt{3} \Rightarrow 4(2)^2 + 16(\sqrt{3})^2 = 2.5.4$$

$16 + 16 \cdot 3 = 64$

Example 9: Find the equation of the tangent line to the ellipse $4x^2 + 16y^2 = 64$ at the point $(2, \sqrt{3})$.

Eqn of tangent
line has
slope $\frac{dy}{dx}$, so
find $\frac{dy}{dx}$

$$\frac{\partial}{\partial x} (4x^2 + 16y^2) = \frac{\partial}{\partial x} (64)$$

$$8x + 32y \frac{dy}{dx} = 0$$

$$32y \frac{dy}{dx} = -8x$$

$$\frac{dy}{dx} = \frac{-8x}{32y} = -\frac{x}{4y}$$

$$\left. \frac{dy}{dx} \right|_{x=2, y=\sqrt{3}} = -\frac{2}{4\sqrt{3}} = -\frac{2}{4\sqrt{3}} \left(\frac{\sqrt{3}}{\sqrt{3}} \right) = -\frac{2\sqrt{3}}{12} = -\frac{\sqrt{3}}{6}$$

$$y - y_1 = m(x - x_1)$$

$y - \sqrt{3} = -\frac{\sqrt{3}}{6}(x - 2)$

$$y - \sqrt{3} = -\frac{\sqrt{3}}{6}x + \frac{2\sqrt{3}}{6}$$

$y = -\frac{\sqrt{3}}{6}x + \frac{\sqrt{3}}{3} + \sqrt{3}$ (2)

$$y = -\frac{\sqrt{3}}{6}x + \frac{4\sqrt{3}}{3}$$

Definition: Two curves are said to be *orthogonal* if at each point of intersection, their tangent lines are perpendicular.

Example 10: Show that the hyperbola $x^2 - y^2 = 5$ and the ellipse $4x^2 + 9y^2 = 72$ are orthogonal.

Ex 12: Find $\frac{dy}{dx}$ for $x^2 + y^2 = 4$

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(4)$$

$$2x + 2y \frac{dy}{dx} = 0$$

Chain rule: $\frac{du}{dx} = \frac{dy}{dx} \cdot \frac{du}{dx}$

Solve for $\frac{dy}{dx}$: $2y \frac{dy}{dx} = -2x$

$$\frac{dy}{dx} = \frac{-2x}{2y}$$

$$\frac{dy}{dx} = -\frac{x}{y}$$

This is the slope of the tangent line at the point (x, y) .

Note:

$$\begin{aligned} \frac{d}{dx}(\sin x)^2 &= 2 \sin x \frac{d}{dx}(\sin x) \\ &= 2 \sin x \cos x \end{aligned}$$

Fails vertical line test, so y is not a function of x .

Solving for y

$$y = \pm \sqrt{4 - x^2}$$

Ex 8 continued:

$$= \frac{-2xy^2 \frac{dy}{dx} + 2y \frac{dy}{dx} + 2y^3}{(2xy-1)^2}$$

Substitute $\frac{dy}{dx} = \frac{-y^2}{2xy-1}$

$$= \frac{-2xy^2 \left(\frac{-y^2}{2xy-1} \right) + 2y \left(\frac{-y^2}{2xy-1} \right) + 2y^3}{(2xy-1)^2}$$

$$\frac{2xy-1}{2xy-1}$$

$$= \frac{2xy^4 - 2y^3 + 2y^3(2xy-1)}{(2xy-1)^3}$$

$$= \frac{2xy^4 - 2y^3 + 4xy^4 - 2y^3}{(2xy-1)^3}$$

$$= \boxed{\frac{6xy^4 - 4y^3}{(2xy-1)^3}}$$

Alternate approach:

$$xy^2 - y = 3$$

$$x \cdot 2y \frac{dy}{dx} + y^2(1) - \frac{dy}{dx} = 0$$

$$2xy \frac{dy}{dx} + y^2 - \frac{dy}{dx} = 0$$

$$2xy \frac{d^2y}{dx^2} + \frac{dy}{dx} \cdot \frac{d}{dx}(2xy) + 2y \frac{dy}{dx} - \frac{d^2y}{dx^2} = 0$$

$$2xy \frac{d^2y}{dx^2} + \frac{dy}{dx} (2x \frac{dy}{dx} + y(2)) + 2y \frac{dy}{dx} - \frac{d^2y}{dx^2} = 0$$

- Then could substitute $\frac{-y^2}{(2xy-1)} = \frac{dy}{dx}$ and

then solve for $\frac{d^2y}{dx^2}$