

3.1: Extrema on an Interval

Absolute maximum and minimum:

If $f(x) \leq f(c)$ for every x in the domain of f , then $f(c)$ is the *maximum*, or *absolute maximum*, of f .

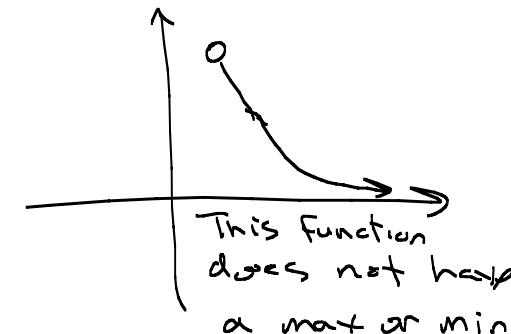
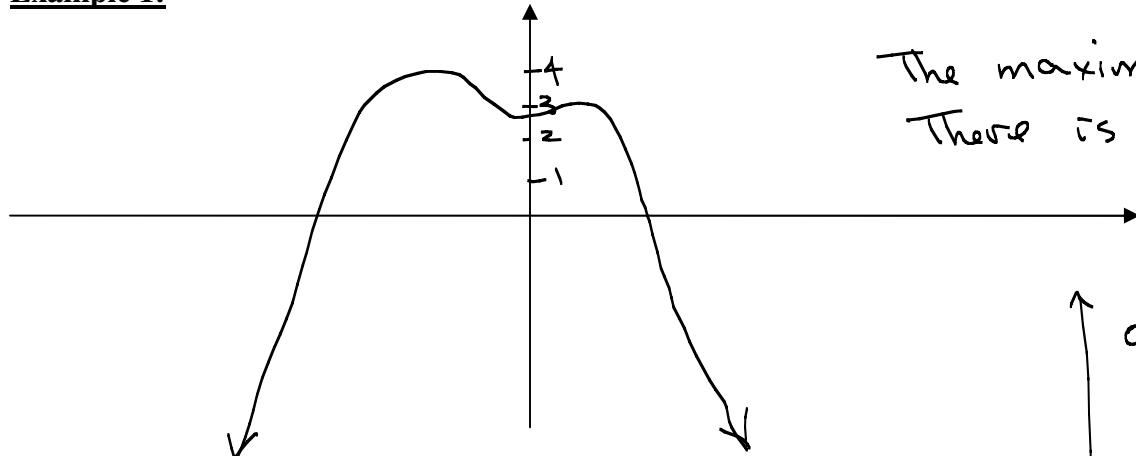
If $f(x) \geq f(c)$ for every x in the domain of f , then $f(c)$ is the *minimum*, or *absolute minimum* of f .

The maximum and minimum values of a function are called the *extreme values* of the function.

In other words, (or maximum, or global maximum)

- The absolute maximum is the largest y -value on the graph.
- The absolute minimum is the smallest y -value on the graph.

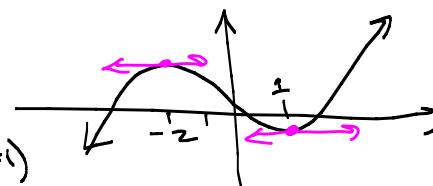
Example 1:



Relative (Local) Maxima and Minima:

- A function f has a *relative maximum*, or *local maximum*, at $x = c$ if there is an interval (a, b) around c such that $f(x) \leq f(c)$ for every x in (a, b) . (These are the “hilltops”).
- A function f has a *relative minimum*, or *local minimum*, at $x = c$ if there is an interval (a, b) around c such that $f(x) \geq f(c)$ for every x in (a, b) . (These are the “bottoms of valleys”).

This function does not have an absolute min or an absolute max. It has a relative maximum at -2 and a relative minimum at 1 . ($x=1$)



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↑
 $(x = -2)$
Notice: If the function is differentiable (smooth), then the tangent line at a local minimum or maximum is horizontal.

so no sharp corner at c

Fermat's Theorem: If f has a local maximum or minimum at c , and if $f'(c)$ exists, then
 $f'(c) = 0$.

(relative)

This means that if f is differentiable at c and has a relative extreme at c , then the tangent line to f at c must be horizontal.

However, we must be careful. The fact that $f'(c) = 0$ (tangent line is horizontal) does not guarantee that there is a relative minimum or maximum at c .

Example 2: $f(x) = x^3$

$$f'(x) = 3x^2$$

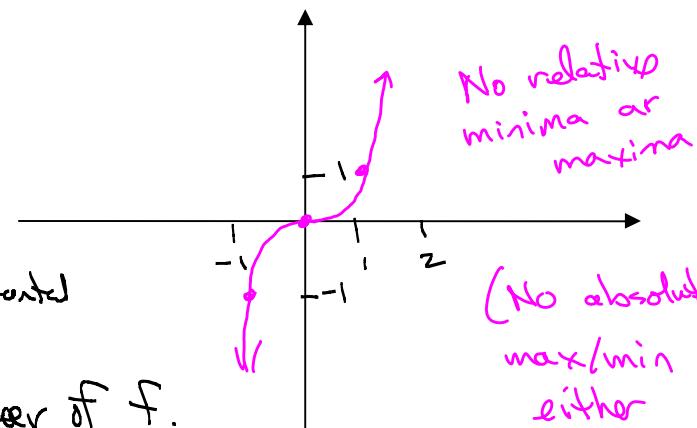
$f'(x) = 0$ when $x = 0$.

$$f'(0) = 3(0)^2 = 0$$

(so tangent line at $x = 0$ is horizontal)

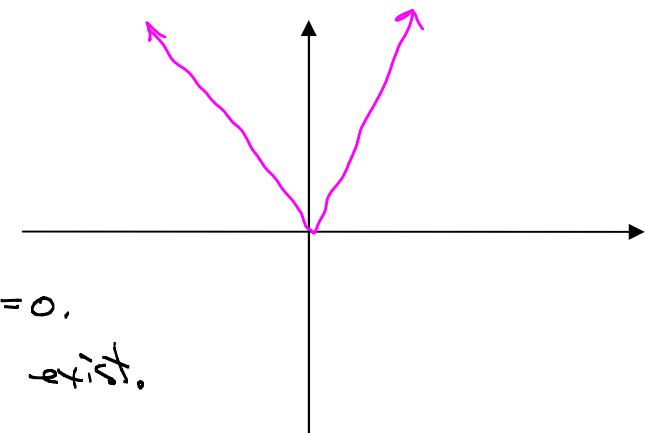
0 is a critical number of f .

(because $f'(0) = 0$.)



Example 3: There can be a local maximum or minimum at c even if $f'(c)$ does not exist.

This function has an absolute min at 0 .
(at the point $(0, 0)$)



Absolute min is $f(0) = 0$.

But $f'(0)$ does not exist.

0 is a critical number
(because f is defined at 0 but $f'(0)$ does not exist)

Critical numbers:

Critical Number: A *critical number* of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.

Theorem: If f has a local maximum or minimum at c , then c is a critical number of f .

(either a sharp corner, or a horizontal tangent)

Note: The converse of this theorem is not true. It is possible for f to have a critical number at c , but not to have a local maximum or minimum at c .

Example 4: Find the critical numbers of $f(x) = x^3 + \frac{17}{2}x^2 - 6x + 4$.

Find $f'(x)$ and set $f'(x) = 0$:

$$\begin{aligned} f'(x) &= 3x^2 + \frac{17}{2}(2x) - 6 \\ &= 3x^2 + 17x - 6 \end{aligned}$$

$$\begin{aligned} \text{set } f'(x) &= 0: \quad 0 = 3x^2 + 17x - 6 \\ &\quad 0 = (3x - 1)(x + 6) \end{aligned}$$

The critical numbers are $-6, \frac{1}{3}$.

Example 5: Find the critical numbers of $f(x) = x^{2/3}$

$$f'(x) = \frac{2}{3}x^{-1/3} = \frac{2}{3\sqrt[3]{x}}$$

$$\begin{aligned} 3x - 1 &= 0 \\ 3x &= 1 \\ x &= \frac{1}{3} \end{aligned}$$

$$\begin{aligned} x + 6 &= 0 \\ x &= -6 \end{aligned}$$

$f'(x)$ is undefined at $x = 0$.

Is f defined at $x = 0$? $f(x) = x^{2/3} = \sqrt[3]{x^2}$. Yes.

So 0 is a critical number.

Example 6: Find the critical numbers of $f(x) = \frac{x^2}{x-3}$

To find critical #s, find $f'(x)$ and set it equal to 0.

$$\begin{aligned} f'(x) &= \frac{(x-3)\frac{d}{dx}(x^2) - x^2\frac{d}{dx}(x-3)}{(x-3)^2} = \frac{(x-3)(2x) - x^2(1)}{(x-3)^2} = \frac{2x^2 - 6x - x^2}{(x-3)^2} \\ &= \frac{x^2 - 6x}{(x-3)^2} = \frac{x(x-6)}{(x-3)^2} \end{aligned}$$

$$\text{Set } f'(x) = 0: \quad 0 = \frac{x(x-6)}{(x-3)^2}$$

$$0 = x(x-6)$$

Critical numbers:
0, 6

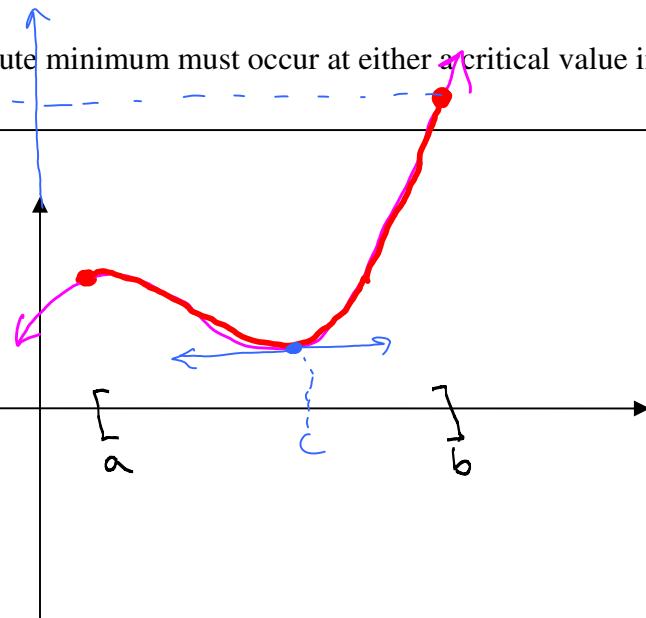
$x = 3$ is not a critical # because 3 is not in domain of f .

Absolute extrema on a closed interval:

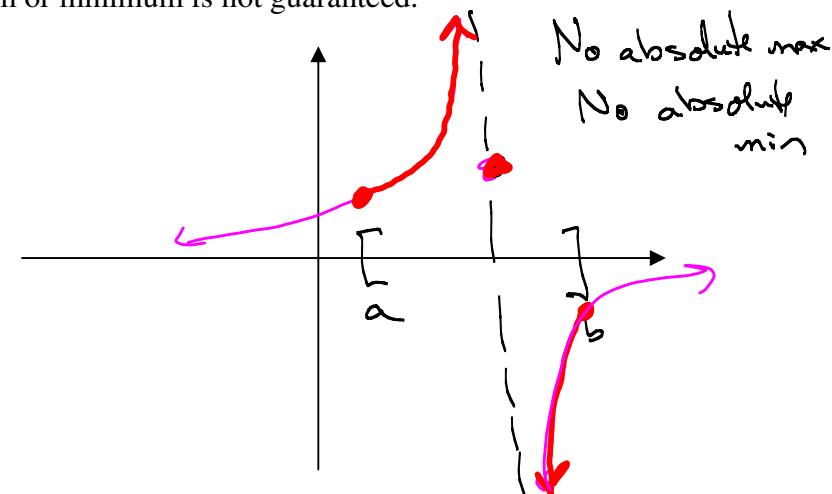
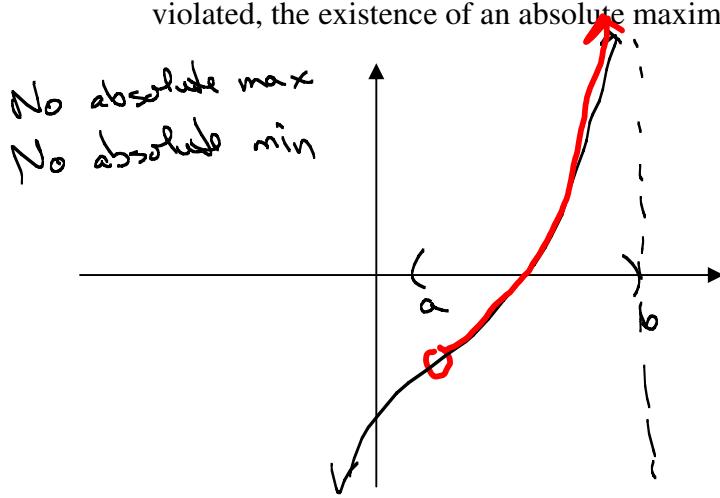
Extreme Value Theorem: If f is continuous on a closed interval $[a,b]$, then f has both an absolute maximum and an absolute minimum on $[a,b]$.

Note: The absolute maximum and the absolute minimum must occur at either a critical value in (a,b) or at an endpoint (at a or b). $f(b)$

Example 7:
 absolute max: $f(b)$ (at endpoint)
 absolute min: $f(c)$
 (at a critical #)



Example 8: If either hypothesis of the extreme value theorem (continuity or closed interval) is violated, the existence of an absolute maximum or minimum is not guaranteed.



Process for Finding the Absolute Extrema of a Continuous Function on a Closed Interval:

1. Find the critical values in (a, b) .
2. Compute the value of f at each critical value in (a, b) and also compute $f(a)$ and $f(b)$.
3. The absolute maximum is the largest of these y -values and the absolute minimum is the smallest of these y -values.

Example 9: Find the absolute extrema for $f(x) = x^2 + 2$ on the interval $[-2, 3]$.

Find critical numbers: $f'(x) = 2x$

Note:
 $x \in [a, b]$ means $a \leq x \leq b$
 $x \in (a, b)$ means $a < x < b$

$$0 = 2x$$

$x=0$ only critical #

$$f(0) = 0^2 + 2 = 2 \quad \text{smallest}$$

$$f(-2) = (-2)^2 + 2 = 4 + 2 = 6$$

$$f(3) = 3^2 + 2 = 9 + 2 = 11 \quad \text{largest}$$

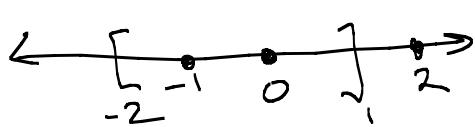
The absolute min is $f(0)=2$.

The absolute max is $f(3)=11$.

Example 10: Find the extreme values of $g(x) = \frac{1}{2}x^4 - \frac{2}{3}x^3 - 2x^2 + 3$ on the interval $[-2, 1]$.

$$g'(x) = \frac{1}{2}(4x^3) - \frac{2}{3}(3x^2) - 4x + 0$$

$$= 2x^3 - 2x^2 - 4x = 2x(x^2 - x - 2) = 2x(x-2)(x+1)$$



Critical #s: 0, 2, -1

Note: 2 is not in my interval $[-2, 1]$

critical #s $\{ g(0) = 3 = \frac{18}{6}$

$$g(-1) = \frac{1}{2} + \frac{2}{3} - 2 + 3 = \frac{3}{6} + \frac{4}{6} + 1 = \frac{13}{6} = \frac{25}{6}$$

endpts $\{ g(-2) = \frac{1}{2}(16) - \frac{2}{3}(-8) - 2(-4)^2 + 3 = 8 + \frac{16}{3} - 8 + 3 = \frac{25}{3}$

$$g(1) = \frac{1}{2} - \frac{2}{3} - 2 + 3 = -\frac{1}{6} + 1 = \frac{5}{6}$$

Absolute max: $g(-2) = \frac{25}{3}$

Absolute min $g(1) = \frac{5}{6}$

$$h(x) = 6\sqrt[3]{x^2}$$

3.1.6

Example 11: Find the absolute extrema of $h(x) = 6x^{2/3}$ on the intervals (a) $[-8, 1]$, (b) $[-8, 1)$, and (c) $(-8, 1)$.

$$h'(x) = 6 \left(\frac{2}{3} x^{-1/3} \right) = 4x^{-1/3} = \frac{4}{\sqrt[3]{x}}$$

h' is undefined at 0. Is h defined at 0? Yes

so 0 is a critical #. h is continuous.

(a) For $[-8, 1]$

$$h(0) = 6\sqrt[3]{0^2} = 0$$

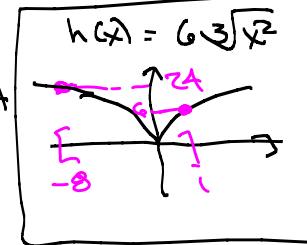
$$h(-8) = 6\sqrt[3]{(-8)^2} = 6\sqrt[3]{64} = h(4) = 24$$

$$h(1) = 6\sqrt[3]{1^2} = 6\sqrt[3]{1} = h(1) = 6$$

absolute min is $h(0) = 0$; abs. max is $h(-8) = 24$

(b) For $[-8, 1)$ abs min is $h(0) = 0$; abs. max is $h(-8) = 24$

(c) For $(-8, 1)$, abs min is $h(0) = 0$; no abs. max



Example 12: Find the absolute maximum and absolute minimum of $f(x) = \sin 2x - x$ on the interval $[0, \pi]$.

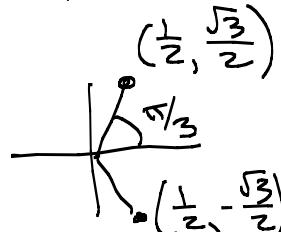
$$f'(x) = (\cos(2x))(2) - 1 = 2\cos 2x - 1$$

Note: f is continuous.

$$\text{Set } f'(x) = 0: 0 = 2\cos 2x - 1$$

$$1 = 2\cos 2x$$

$$\frac{1}{2} = \cos 2x$$



$$0 \leq 2x \leq 2\pi$$

$$0 \leq x \leq \pi$$

$$f\left(\frac{\pi}{6}\right) = \sin\left(\frac{2\pi}{6}\right) - \frac{\pi}{6} = \frac{\sqrt{3}}{2} - \frac{\pi}{6} \approx 0.34241$$

$$f\left(\frac{5\pi}{6}\right) = \sin\left(\frac{5\pi}{3}\right) - \frac{5\pi}{6} = -\frac{\sqrt{3}}{2} - \frac{5\pi}{6} \approx -3.48402$$

$$f(0) = \sin(2 \cdot 0) - 0 = \sin 0 - 0 = 0$$

$$f(\pi) = \sin(2\pi) - \pi = 0 - \pi = -\pi \approx -3.14159$$

Absolute max is $f\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} - \frac{\pi}{6}$

Absolute min is $f\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2} - \frac{5\pi}{6}$