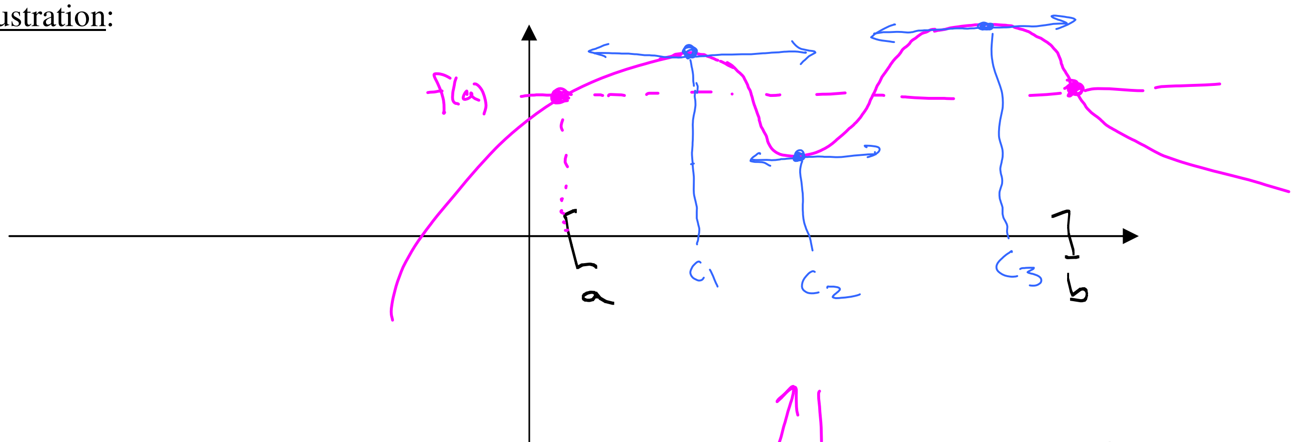
3.2: Rolle's Theorem and the Mean Value Theorem

<u>Rolle's Theorem</u>:

Let f be a function that is continuous on the closed interval $[a,b]$ and differentiable on the open interval (a,b).

If $f(a) = f(b)$, then there is a number c in (a,b) such that $f'(c) = 0$.

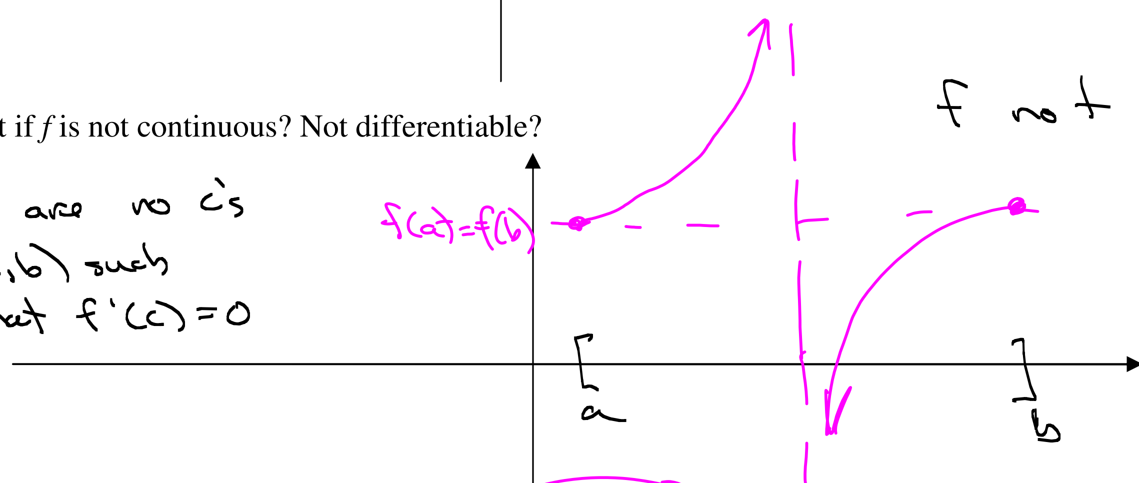
<u>Illustration</u>:



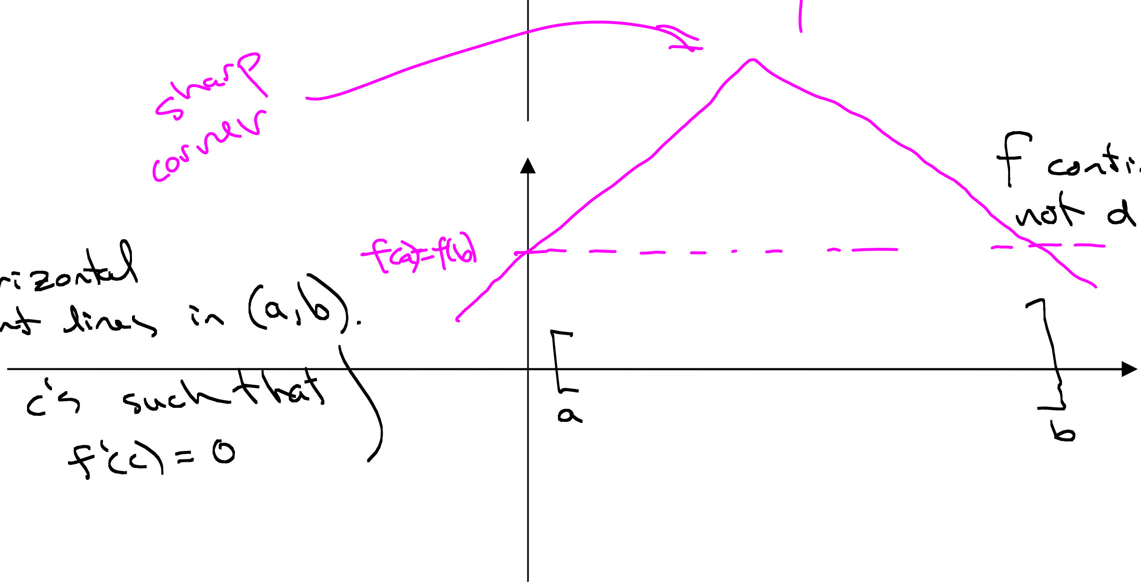
What if f is not continuous? Not differentiable?

f not continuous

There are no c's in (a,b) such that f'(c) = 0



sharp corner



f continuous but not differentiable

No horizontal tangent lines in (a,b).
(so no c's such that f'(c) = 0)

Example 1: Show that the function $f(x) = x^2 - 4x - 5$ satisfies the hypotheses of Rolle's Theorem on the interval $[-1,5]$. Find all numbers c in $[-1,5]$ that satisfy the conclusion of Rolle's Theorem.

f is a polynomial, so it is continuous and differentiable on $(-\infty, \infty)$.

$f(-1) = (-1)^2 - 4(-1) - 5 = 1 + 4 - 5 = 0$

$f(5) = (5)^2 - 4(5) - 5 = 25 - 25 = 0$

So $f(-1) = f(5)$.

Find $f'(x)$ and set it $= 0$: $f'(x) = 2x - 4$

$0 = 2x - 4$

$0 = 2(x-2)$

$x = 2$

2 is the only c in $[-1,5]$ such that $f'(c) = 0$.

Example 2: Show that the function $g(x) = -2x^4 + 16x^2$ satisfies the hypotheses of Rolle's Theorem on the interval $[-3,3]$. Find all numbers c that satisfy the conclusion of Rolle's Theorem.

Continuous & differentiable? Yes, it's a polynomial.

$g(-3) = f(3)$.

$g'(x) = -8x^3 + 32x$

$= -8x(x^2 - 4) = -8x(x+2)(x-2)$

Critical #s are $\boxed{0, 2, -2}$.

All three are in $[-3,3]$ and have $f'(c) = 0$.

<u>Mean Value Theorem</u>:

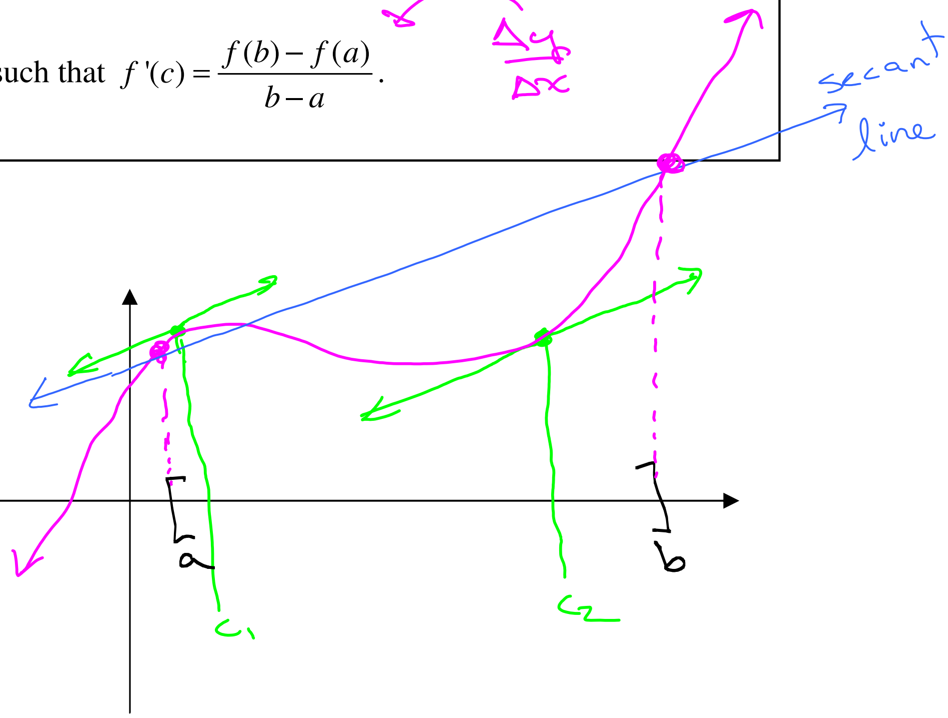
Let f be a function that is continuous on the closed interval $[a,b]$ and differentiable on the open interval (a,b).

Then there is a number c in (a,b) such that $f'(c) = \dfrac{f(b) - f(a)}{b-a}$.

$\frac{\Delta y}{\Delta x}$

<u>Illustration</u>:

There must be a c where the tangent line is parallel to the secant line (from (a,f(a)) to (b,f(b)))



A few consequences of the Mean Value Theorem:

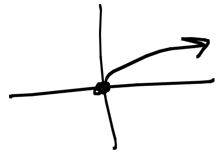
1) The Mean Value Theorem guarantees the existence of a tangent line parallel to the secant line that contains the endpoints $(a, f(a))$ and $(b, f(b))$.

2) In terms of rates of change, the Mean Value Theorem guarantees that there is some point at which the <u>instantaneous</u> rate of change is equal to the <u>average</u> rate of change over $[a,b]$.

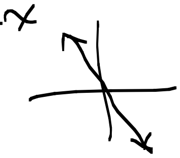
3) If the derivative of a function is 0 for every number in an interval, then the function is constant on that interval.

4) If two functions have the same derivative on an interval, they differ by a constant on that interval.

$y = \sqrt{x}$



$y = -2x$



Example 3: Verify that the function satisfies the hypotheses of the Mean Value Theorem on the given interval. Then find all numbers c that satisfy the conclusion of the Mean Value Theorem.

$f(x) = \sqrt{x} - 2x$ on the interval $[0,4]$

Domain: $[0, \infty)$

Continuous and differentiable on $[0,\infty)$.

Find slope of secant line:

$$m = \frac{\Delta y}{\Delta x} = \frac{f(4) - f(0)}{4-0} = \frac{-6-0}{4-0} = -\frac{6}{4} = -\frac{3}{2}$$

Find points: $f(0) = \sqrt{0} - 2(0) = 0 \Rightarrow (0,0)$

$f(4) = \sqrt{4} - 2(4) = 2 - 8 = -6 \Rightarrow (4,-6)$

Find $f'(x)$ and set it equal to $-\frac{3}{2}$:

$f(x) = x^{1/2} - 2x$

$f'(x) = \frac{1}{2}x^{-1/2} - 2 = \frac{1}{2\sqrt{x}} - 2$

$\frac{1}{2\sqrt{x}} - 2 = -\frac{3}{2}$

multiply by $2\sqrt{x}$:

$\frac{1}{2\sqrt{x}}(2\sqrt{x}) - 2(2\sqrt{x}) = -\frac{3}{2}(2\sqrt{x})$

$1 - 4\sqrt{x} = -3\sqrt{x}$

$1 = 4\sqrt{x} - 3\sqrt{x}$

$1 = \sqrt{x}$

$(1)^2 = (\sqrt{x})^2$

$1 = x$

So $x=1$ is the only c value that satisfies the conclusion of the MVT.

Example 4: As you drive by a Houston police car, your speed is clocked at 50 miles per hour. Five minutes and five miles further, you pass by another HPD officer, who clocks you at 55 miles per hour. If the speed limit on the entire stretch of highway is 55 mph, would the second officer be justified in writing you a speeding ticket?

Slope of secant line = average rate of change

Slope of tangent line = instantaneous rate of change

Average rate of change = Average velocity $= \frac{\Delta \text{distance}}{\Delta \text{time}} = \frac{5 \text{ miles}}{5 \text{ minutes}}$

$= \frac{1 \text{ miles}}{\text{min}}\left(\frac{60 \text{ min}}{1 \text{ hr}}\right)$

$= 60$ miles/hr

Because of Mean Value Theorem, somewhere between the officers, I had to be driving 60 mph at some point.

So yes, he can write a ticket!