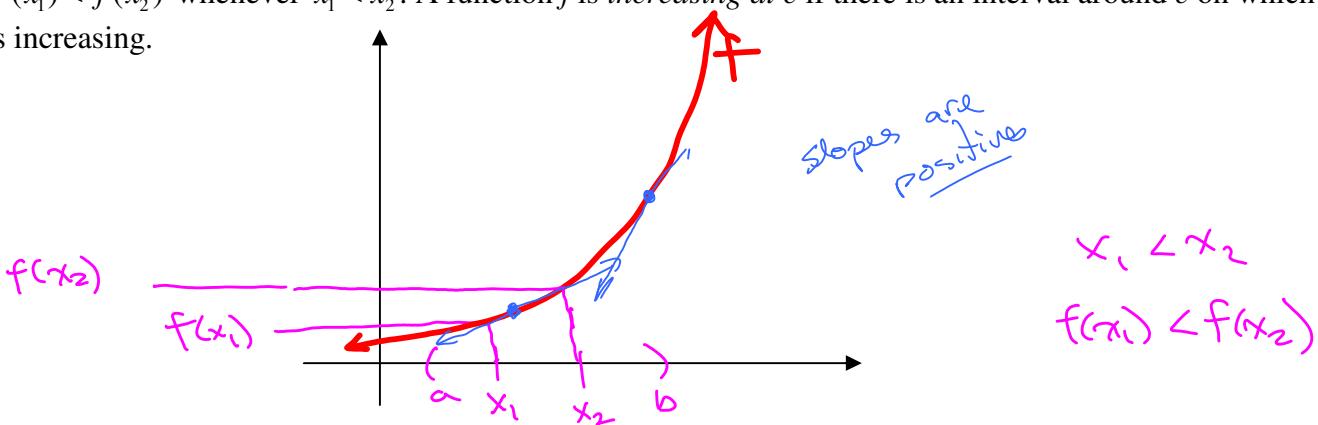


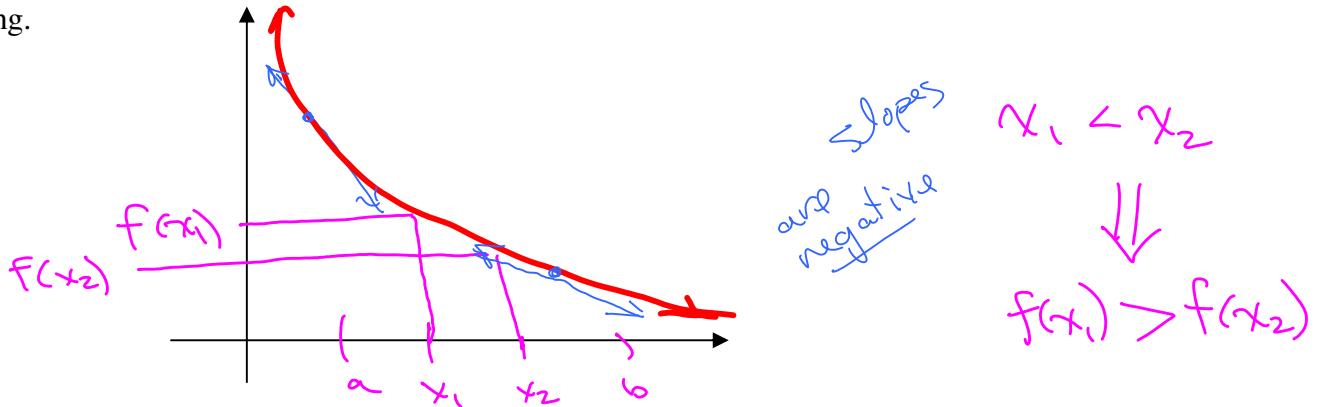
3.3: Increasing and Decreasing Functions and the First Derivative Test

Increasing and decreasing functions:

A function f is said to be *increasing* on the interval (a, b) if, for any two numbers x_1 and x_2 in (a, b) , $f(x_1) < f(x_2)$ whenever $x_1 < x_2$. A function f is *increasing at c* if there is an interval around c on which f is increasing.



A function f is said to be *decreasing* on the interval (a, b) if, for any two numbers x_1 and x_2 in (a, b) , $f(x_1) > f(x_2)$ whenever $x_1 < x_2$. A function f is *decreasing at c* if there is an interval around c on which f is decreasing.



Notice that wherever a function is increasing, the tangent lines have positive slope.
Notice that wherever a function is decreasing, the tangent lines have negative slope.

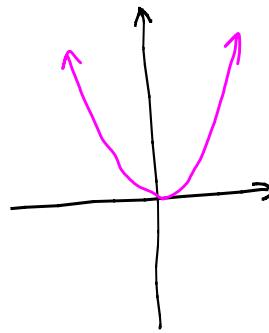
This means that we can use the derivative to determine the intervals where a function is increasing and decreasing.

Increasing/Decreasing Test: Let f be a function that is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b) .

- If $f'(x) > 0$ for every x in (a, b) , then f is increasing on (a, b) .
- If $f'(x) < 0$ for every x in (a, b) , then f is decreasing on (a, b) .
- If $f'(x) = 0$ for every x in (a, b) , then f is constant on (a, b) .

Example 1: $f(x) = x^2$

Increasing on $(0, \infty)$
 Decreasing on $(-\infty, 0)$



Note: $f'(x) = 2x$

For $x < 0$, $f'(x) = 2x < 0$

For $x > 0$, $f'(x) = 2x > 0$

Steps for Determining Increasing/Decreasing Intervals

- Find all the values of x where $f'(x) = 0$ or where $f'(x)$ is not defined. Use these values to split the number line into intervals.
 - Choose a test number c in each interval and determine the sign of $f'(c)$.
 - If $f'(c) > 0$, then f is increasing on that interval.
 - If $f'(c) < 0$, then f is decreasing on that interval.
- $f'(c) > 0 \Rightarrow f \text{ is increasing}$

Note: Three types of numbers can appear on your number line:

- Numbers where the function is defined and the derivative is 0. (These are critical numbers.)
- Numbers where the function is defined and the derivative is undefined. (These are also critical numbers.)
- Numbers where the function is undefined. (These are NOT critical numbers.)

First derivative test:

This procedure determines the relative extrema of a function f .

First derivative test:

Suppose that c is a critical number of a function f that is continuous on an open interval containing c .

- If $f'(x)$ changes from positive to negative across c , then f has a relative maximum at c .
- If $f'(x)$ changes from negative to positive across c , then f has a relative minimum at c .
- If $f'(x)$ does not change sign across c , then f does not have a relative extreme at c .



(Example 2 is continued on last page of file)

3.3.3

Example 2: Determine the intervals on which $f(x) = x^3 + 6x^2 - 36x + 18$ is increasing and decreasing.

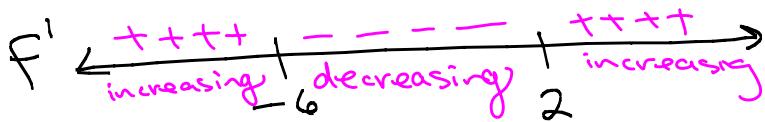
Find the relative extrema. Domain: $(-\infty, \infty)$

f is increasing on $(-\infty, -6)$ and on $(2, \infty)$,
 f is decreasing on $(-6, 2)$.

$$\begin{aligned} f'(x) &= 3x^2 + 12x - 36 \\ &= 3(x^2 + 4x - 12) \\ &= 3(x+6)(x-2) \end{aligned}$$

$$f'(x) = 0 \Rightarrow 0 = 3(x+6)(x-2)$$

Critical #s: $-6, 2$



$(-\infty, -6)$: Test number $x = -7$

$$\begin{aligned} f'(-7) &= 3(-7)^2 + 12(-7) - 36 \\ &= 3(x+6)(x-2) \end{aligned}$$

$$f'(-7) = 3(-7)^2 + 12(-7) - 36 = 27 \quad (+)$$

$$\begin{aligned} f'(-7) &= 3(-7+6)(-7-2) \\ &= 3(-1)(-9) \\ &= 27 \quad (+) \end{aligned}$$

$(2, \infty)$: $x = 12$

$$\begin{aligned} f'(12) &= 3(12+6)(12-2) \\ &= 3(18)(10) \\ &\Rightarrow (+) \end{aligned}$$

Example 3: Determine the intervals on which $g(x) = x^3 - 6x^2 + 12x - 8$ is increasing and decreasing.

Find the relative extrema.

$$g'(x) = 3x^2 - 12x + 12$$

$$= 3(x^2 - 4x + 4)$$

$$= 3(x-2)^2$$

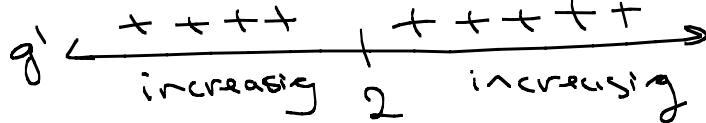
Critical #: 2

$(-\infty, 2)$: Test $x = 0$

$$\begin{aligned} g'(0) &= 3(0-2)^2 \\ &= 3(-)^2 \end{aligned}$$

$$= (+)(+) \Rightarrow (+)$$

g is increasing on $(-\infty, \infty)$
No relative extrema



$(2, \infty)$: Test $x = 3$

$$\begin{aligned} g'(3) &= 3(3-2)^2 \\ &= 3(+)^2 \end{aligned}$$

Example 4: Determine the intervals on which $g(x) = x^{\frac{2}{5}}$ is increasing and decreasing. Find the relative extrema.

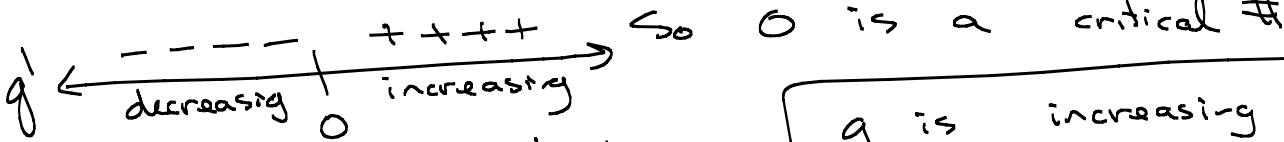
$$g(x) = x^{\frac{2}{5}} = \sqrt[5]{x^2} \quad \text{Domain: } (-\infty, \infty)$$

$$g'(x) = \frac{2}{5} x^{-\frac{3}{5}} = \frac{2}{5\sqrt[5]{x^3}}$$

relative min at $x=0$

where is $g'(x)=0$? Nowhere

where is $g'(x)$ undefined? at $x=0$



Find g -value at local min:
 $g(0) = \sqrt[5]{0^2} = 0$

Relative minimum is $g(0) = 0$.

g is increasing on $(0, \infty)$
 and decreasing on $(-\infty, 0)$.

(Note: this is also the absolute min.)

Example 5: Determine the intervals on which $f(x) = x + \frac{4}{x}$ is increasing and decreasing. Find the relative extrema.

Domain of f : $x \neq 0$
 $(-\infty, 0) \cup (0, \infty)$

Critical numbers: -2, 2
 0 is not a critical # because
 0 is not in domain of f

$$f(x) = x + 4x^{-1}$$

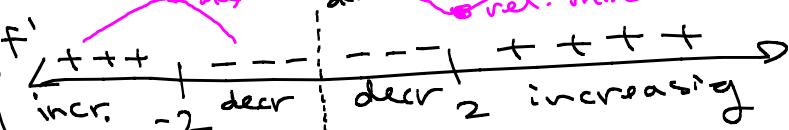
$$f'(x) = 1 - 4x^{-2} = 1 - \frac{4}{x^2}$$

$$= \frac{x^2 - 4}{x^2} = \frac{(x+2)(x-2)}{x^2}$$

$f'(x)$ is undefined for $x=0$

$f'(x) = 0$ for $x=-2, x=2$

$$0 = \frac{(x+2)(x-2)}{x^2} \Rightarrow 0 = (x+2)(x-2) \Rightarrow x = \pm 2$$



f is increasing on $(-\infty, -2)$ and on $(2, \infty)$.

f is decreasing on $(-2, 0)$ and on $(0, 2)$.

f has a relative max at $(-2, -4)$, and a relative min at $(2, 4)$.

$$f'(x) = \frac{(-x+2)(-x-2)}{x^2}$$

$(-\infty, -2)$: Test $x = -3$

$$f'(-3) = \frac{(-3+2)(-3-2)}{(-3)^2} \Rightarrow \frac{(-)(-)}{(+)} \Rightarrow \frac{(+)}{(+)}$$

because $0 = \frac{(x+2)(x-2)}{x^2} \Rightarrow 0 = (x+2)(x-2)$
 $x = \pm 2$

$(-2, 0)$: Test $x = -1$

$$f'(-1) = \frac{(-1+2)(-1-2)}{(-1)^2} \Rightarrow \frac{(+)(-)}{(+)} \Rightarrow (-)$$

$(0, 2)$: Test $x = 1$

$$f'(1) = \frac{(1+2)(1-2)}{1^2} \Rightarrow \frac{(+)(-)}{(+)} \Rightarrow (-)$$

$$(2, \infty) \Rightarrow \frac{(+)(+)}{(+)^2} \Rightarrow (+)$$

Ex 5: Find y-values: $f(-2) = -2 + \frac{4}{-2} = -2 - 2 = -4$

$$f(2) = 2 + \frac{4}{2} = 2 + 2 = 4$$

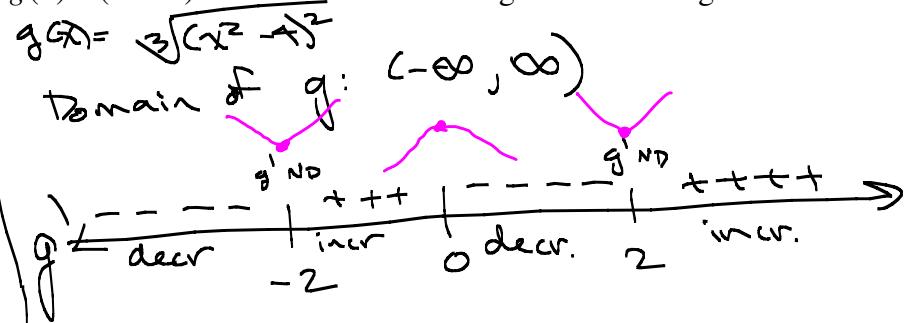
3.3.5

Example 6: Find the local extremes of $g(x) = (x^2 - 4)^{\frac{1}{3}}$. Where is it increasing and decreasing?

$$\begin{aligned} g'(x) &= \frac{2}{3}(x^2 - 4)^{\frac{-2}{3}}(2x) \\ &= \frac{2}{3} \cdot \frac{1}{\sqrt[3]{x^2 - 4}} \cdot 2x \\ &= \frac{4x}{3\sqrt[3]{x^2 - 4}} \end{aligned}$$

g' undefined at $x = \pm 2$
 $g' = 0$ at $x = 0$.

g is increasing on $(-2, 0)$ and on $(2, \infty)$.
 g is decreasing on $(-\infty, -2)$ and on $(0, 2)$.
 Relative minima at $x = \pm 2$
 Relative maximum at $x = 0$.



$$(-\infty, -2): x = -3$$

$$g'(-3) = \frac{4(-3)}{3\sqrt[3]{(-3)^2 - 4}} \Rightarrow \frac{(-)}{3\sqrt[3]{(+)}} \Rightarrow (-)$$

$$(-2, 0): x = -1 \Rightarrow$$

$$g'(-1) = \frac{4(-1)}{3\sqrt[3]{(-1)^2 - 4}} \Rightarrow \frac{(-)}{3\sqrt[3]{(-)}} \Rightarrow \frac{(-)}{(-)} \Rightarrow (+)$$

$$(0, 2): x = 1 \Rightarrow$$

$$g'(1) = \frac{4(1)}{3\sqrt[3]{-4}} \Rightarrow \frac{(+)}{(-)} \Rightarrow (-)$$

$$(2, \infty): x = 3 \Rightarrow g'(3) = \frac{4(3)}{3\sqrt[3]{3^2 - 4}}$$

Example 7: Find the relative extremes of $f(x) = \frac{1}{2}x - \sin x$ on the interval $(0, 2\pi)$. Where is it increasing and decreasing on that interval? $\Rightarrow \frac{(+)}{(-)}$

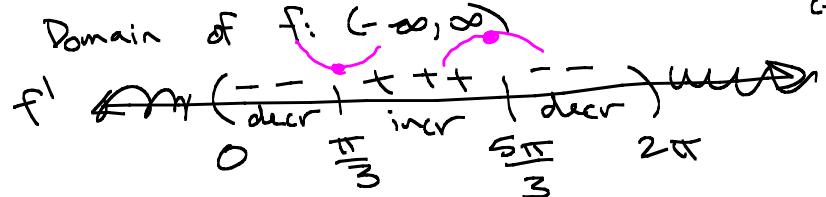
$$f'(x) = \frac{1}{2} - \cos x$$

$$\text{Set } f'(x) = 0 : 0 = \frac{1}{2} - \cos x$$

$$\frac{1}{2} = \cos x$$

$$x = \frac{\pi}{3}, \frac{5\pi}{3}$$

$$\text{Critical #s: } \frac{\pi}{3}, \frac{5\pi}{3}$$



$$(0, \frac{\pi}{3}): \text{Test } x = \frac{\pi}{6}$$

$$\begin{aligned} f'(\frac{\pi}{6}) &= \frac{1}{2} - \cos \frac{\pi}{6} \\ &= \frac{1}{2} - \frac{\sqrt{3}}{2} \Rightarrow (-) \end{aligned}$$

$$(\frac{\pi}{3}, \frac{5\pi}{3}) \text{ Test } x = \pi$$

$$\begin{aligned} f'(\pi) &= \frac{1}{2} - \cos \pi = \frac{1}{2} - (-1) \\ &= \frac{1}{2} + 1 = \frac{3}{2} \quad (+) \end{aligned}$$

$$(\frac{5\pi}{3}, 2\pi) \text{ Test } x = \frac{11\pi}{6}$$

$$f'(\frac{11\pi}{6}) = \frac{1}{2} - \cos \left(\frac{11\pi}{6}\right) = \frac{1}{2} - \frac{\sqrt{3}}{2} \Rightarrow (-)$$

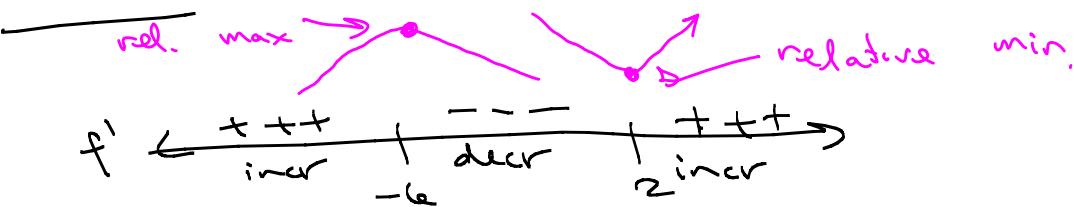
Find y-values of extrema:

$$f(\frac{\pi}{3}) = \frac{1}{2}(\frac{\pi}{3}) - \sin \frac{\pi}{3} = \frac{\pi}{6} - \frac{\sqrt{3}}{2}$$

$$f(\frac{5\pi}{3}) = \frac{1}{2}(\frac{5\pi}{3}) - \sin \frac{5\pi}{3} = \frac{5\pi}{6} - (-\frac{\sqrt{3}}{2}) = \frac{5\pi}{6} + \frac{\sqrt{3}}{2}$$

Relative min is $f(\frac{\pi}{3}) = \frac{\pi}{6} - \frac{\sqrt{3}}{2}$
 Relative max is $f(\frac{5\pi}{3}) = \frac{5\pi}{6} + \frac{\sqrt{3}}{2}$

Ex 2 cont'd Find the relative extrema



f has a relative maximum at $x = -6$
and a relative minimum at $x = 2$.

Find the y-values:

$$f(x) = x^3 + 6x^2 - 36x + 18$$

$$\begin{aligned}f(-6) &= (-6)^3 + 6(-6)^2 - 36(-6) + 18 \\&= -216 + 216 + 216 + 18 = 234\end{aligned}$$

$$\begin{aligned}f(2) &= 2^3 + 6(2)^2 - 36(2) + 18 \\&= 8 + 24 - 72 + 18 = 50 - 72 = -22\end{aligned}$$

Relative max: $f(-6) = 234$

Relative min: $f(2) = -22$